11:58:08 OCA PAD AMENDMENT - PROJECT HEADER INFORMATION 09/23/91

Active

Project #: E-24-672 Cost share #: E-24-371 Rev #: 6
Center # : R6722-0A0 Center shr #: F6722-0A0 OCA file #:
Work type : RES
Contract#: SES-8821999 Mod #: AMENDMENT #2 Document : AGR
Prime #: Contract entity: GTRC

Subprojects ? : N CFDA:
Main project #: PE #:

Project unit: ISYE Unit code: 02.010.124
Project director(s):
FOLEY R D ISYE (404)894-2309
FRAZELLE E H ISYE (404)-

Sponsor/division names: NATL SCIENCE FOUNDATION / GENERAL
Sponsor/division codes: 107 / 000

Award period: 890501 to 920131 (performance) 920430 (reports)

Sponsor amount	New this change	Total to date
Contract value	0.00	133,700.00
Funded	0.00	133,700.00
Cost sharing amount		3,200.00

Does subcontracting plan apply ?: N

Title: ANALYTICAL RESULTS FOR MATERIAL HANDLING SYSTEMS

PROJECT ADMINISTRATION DATA

OCA contact: Mildred S. Heyser 894-4820

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Security class (U,C,S,TS) : U ONR resident rep. is ACO (Y/N): N
Defense priority rating : N/A NSF supplemental sheet
Equipment title vests with: Sponsor GIT X

Administrative comments -
PERFORMANCE ENDING DATE EXTENDED TO 1/31/92 VIA OPAS.

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT

27 N- 5R 342

Closeout Notice Date 08/26/92

Project No. E-24-672 Center No. R6722-0A0

Project Director FOLEY R D School/Lab ISYE

Sponsor NATL SCIENCE FOUNDATION/GENERAL

Contract/Grant No. SES-8821999 Contract Entity GTRC

Prime Contract No.

Title ANALYTICAL RESULTS FOR MATERIAL HANDLING SYSTEMS

Effective Completion Date 920131 (Performance) 920430 (Reports)

Closeout Actions Required:	Y/N	Date Submitted
Final Invoice or Copy of Final Invoice	N	
Final Report of Inventions and/or Subcontracts	N	
Government Property Inventory & Related Certificate	N	
Classified Material Certificate	N	
Release and Assignment	N	
Other	N	

CommentsBILLING VIA LINE OF CREDIT

Subproject Under Main Project No.

Continues Project No.

Distribution Required:

Project Director	Y
Administrative Network Representative	Y
GTRI Accounting/Grants and Contracts	Y
Procurement/Supply Services	Y
Research Property Managment	Y
Research Security Services	N
Reports Coordinator (OCA)	Y
GTRC	Y
Project File	Y
Other	N
	N

E-24-672

Progress Report

NSF Grant No.: SES-8821999

Title: Analytical Results for Material Handling Systems

Principal Investigators: Robert D. Foley and Edward H. Frazelle

Period: 5/1/89 to 4/31/90, first year of a two-year continuing grant

1 Scientific Progress

1.1 Work Completed

The following results have been established and papers are either in preparation or available.

- Throughput expressions for miniload systems with *square-in-time racks* [1]
- Analytical expressions and bounds for miniload throughput operated under *class-based storage* [2]
- Throughput expressions for *rectangular-in-time* miniload systems with random storage [3]
- Asymptotically efficient storage policies for shared, class-based storage systems [4]

1.2 Work in Progress

- Empirical validation of throughput expressions
- Miniload retrieval sequencing policies
- Analysis of remote order picking systems

2 Technology Transfer

Since we want our results to be used, we have made an effort to present our results in several media in addition to scientific journals.

2.1 Presentations

- AS/RS Users Association Annual Conference (twice)
- Material Handling Research Center Monitors Meetings
- Material Handling Institute Teachers Conference
- Material Handling Research Colloquium

2.2 Software

To make our our analytical results more accesible, we implemented them results in a software package. The user inputs information about the miniload system through a graphical interface. The proposed system is analyzed and several system performance measures are presented.

3 Current Personnel

Principal Investigators: Robert D. Foley and Edward H. Frazelle

Ph.D. Students: S. Lim (just completed), J. Park, and B. Park (funded from another source)

Master's Student: D. Scharfstein

4 Budget

We anticipate that the first year's budget will be completely spent by 4/31/90.

References

[1] Foley, R. D. and E. H. Frazelle, "Analytical Results for Miniload Throughput and the Distribution of Dual Command Travel Time," to appear *IIE Transactions*.

[2] Foley, R. D. and E. H. Frazelle, "Throughput Expressions and Bounds for End-of-Aisle Order Picking with Activity Based Storage," draft attached.

[3] Scharfstein, D., Master's Thesis in preparation.

[4] Lim S., "Zoning in Storage Systems," Ph.D. Thesis, abstract attached.

Zoning in Shared Storage Systems

Sanggyu Lim

February 1990

Abstract

Suppose that two types of items with different turnover rates (high and low) are stored in the same storage rack. Then in the COL (Closest Open Location) Storage or Random Storage policy, high turnover items do not have priority.

In this dissertation, two storage policies, Zone-z Storage and Skip-k Storage, which give priority to high turnover items over low turnover items, are addressed. In both storage policies, high turnover items are stored in the COL. In Zone-z Storage, low turnover items are stored in the COL after location z and in Skip-k Storage, stored in the $(k+1)^{st}$ open location. Hence high turnover items will have a better chance of being stored in the closer locations. Hence these two storage policies will induce favorable zoning and reduce the average storage and retrieval distance.

It is assumed that type $h[l]$ items arrive at the storage rack according to a Poission process with rate $\lambda_h[\lambda_l]$ and each arrived type $h[l]$ item is stored for an exponentially distributed length of time with parameter $\mu_h[\mu_l]$, respectively. We will like to find values of z and k in terms of the system parameters, which will minimize expected travel distance.

We defined a desirable property of a good policy which we call *asymptotically efficient*, and show that the Zone-z policy is asymptotically efficient if the zone size is proportional to the expected number of high turnover items in the system in equilibrium. We extend some of the results to compound Poisson arrival processes, i.e., batch arrivals. A conjectured asymptotically efficient skip size is also given along with supporting evidence. Numerical examples show that the asymptotically efficient zone size or skip size are close to the optimal ones as obtained from simulation.

On the other hand, it is shown that the COL Storage policy is not asymptotically efficient and that asymptotically efficient policies will provide a substantial decrease in travel time. The basic results are also extended to square storage rack with Tchebychev travel metric.

Analytical Results for Miniload Throughput and the Distribution of Dual Command Travel Time

R. D. Foley E. H. Frazelle

August 21, 1988
Revised September 20, 1989

Abstract

We determine the throughput, the maximum rate at which a miniload automated storage and retrieval system can process requests, as a function of the system parameters such as the dimension of the storage racks, the speed of the s/r machine, and the speed of the picker. As a byproduct, we determine the distribution function of the travel time for a storage/retrieval machine that stores one container and retrieves another container ("dual command cycle") before returning to the I/O point. Our emphasis is on obtaining closed-form, computable expressions.

1 Introduction

Miniload automated storage and retrieval systems (AS/RS) are popular for small parts order picking because they provide excellent space utilization, excellent item security, and accurate item picking [8]. In fact, over 500 miniload systems are currently in use in the U.S. [14]. If designed and operated properly, miniload systems can be reliable and cost effective for small parts order picking. However, at $300,000 per aisle [18] and with limited ability to reconfigure, the initial system design must be accurate. Thus, analytical results, in particular closed-form computable expressions, for determining the system performance would be extremely useful for designing the system. Combined with cost information, analytical results would permit the rapid exploration of many potential system configurations.

Toward that end, we analytically derive several system performance measures including system throughput (the maximum rate at which the miniload and picker can process containers), the utilization of the picker, the utilization of the s/r machine, and the distribution of dual command travel time.

Section 2 describes miniload order picking and the assumptions of our model. Section 3 reviews the literature concerning analytical models of AS/RS performance. In Section 4, we derive expressions for the system throughput. In order to use these expressions, we need to derive the distribution of s/r machine travel time which is done for square-in-time racks in Section 5. Sections 6 and 7 contain examples of how the results can be used. The last section summarizes our paper and points out possible extensions.

2 Miniload AS/RS and Model Assumptions

Miniload automated storage and retrieval systems (AS/RS) are in frequent use to store and retrieve items for small parts order picking. A typical system, see Figure 1, is comprised of multiple aisles of storage rack, a storage/retrieval (s/r) machine operating in each aisle, numerous modular storage containers for housing the items, and load stands at the end of each aisle to facilitate order picking.

The load stands are arranged such that each aisle has a left and a right pick position. While the order picker is extracting items from the container in one pick position, the s/r machine returns the container in the other pick position to its home location in the rack, and returns with the next container for picking. Systems may be configured with more than two pick positions per aisle, with a conveyor delivery system to deliver containers to "remote" order pickers, or with multiple input/output (I/O) points per aisle. However, it is this typical configuration [17, pages 9-2 and 9-7], with one picker and two pick positions per aisle, and one I/O point per aisle, that is modeled here. Since each aisle operates independently, we will limit our analysis to one aisle.

The s/r machine travels in an aisle between two rectangular storage racks. The s/r machine requires one unit of time to travel the length of the aisle from the front to the back, and b units of time to travel from the top to the bottom of the rack. The bottom front of the rack will be denoted by the cartesian coordinates $(0,0)$ and the top rear by $(1,b)$; we will not need to differentiate between the right and left rack. The s/r machine retrieves a container from a location in the rack and places it on the empty pick-position.

As mentioned, we assume that there are two pick-positions and one person picking at the end of the aisle. If or when the container on one of the pick-positions has been processed, the s/r machine picks up the container and returns it to its home location. Note that each container has a permanent home in the rack, i.e., dedicated storage.

Let L_n and P_n denote the location and pick-time associated with the nth request, respectively. Since we are interested in the maximum throughput, we assume that there is an infinite queue of requests to be processed and that they are processed in the order of arrival (FCFS). When the person or picker begins to pick items from the container in location L_n the s/r machine leaves to

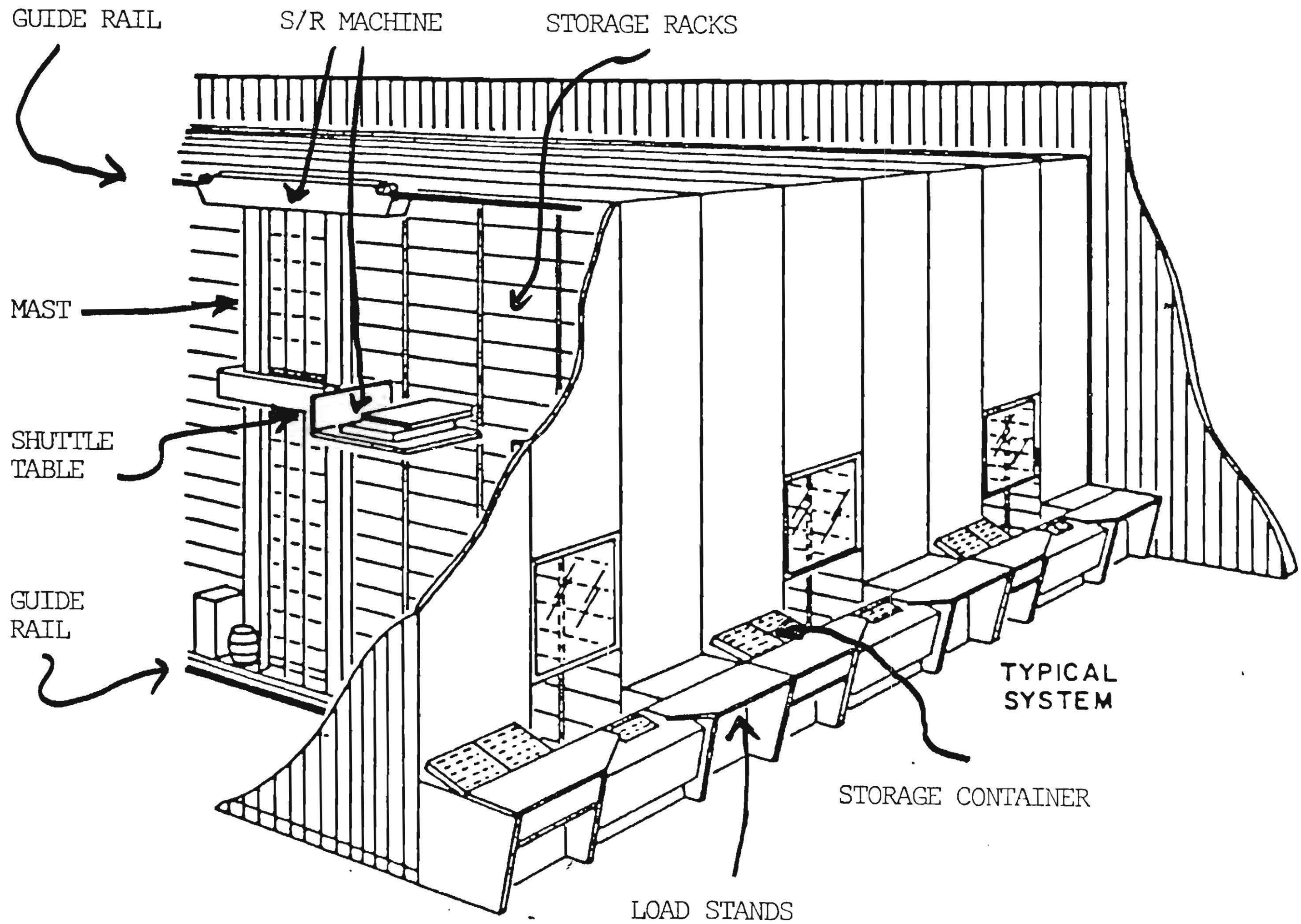

Figure 1: Miniload AS/RS, reprinted from [17].

return the finished container from location L_{n-1} and bring the next container from location L_{n+1}. When finished with a container, the picker informs the s/r machine that the container needs to be returned. If the container from location L_{n+1} is on the other pick-station, the picker will start working on that container. Otherwise, the picker must wait for the s/r machine to bring the container from L_{n+1}.

Let C_n denote the length of time needed by the s/r machine to store the container from location L_{n-1} and retrieve the container from location L_{n+1}. We will assume that C_n can be expressed as

$$C_n = ||L_{n-1}|| + ||L_{n-1} - L_{n+1}|| + ||L_{n+1}|| + c \tag{1}$$

where $||x - y||$ is both the distance and the length of time needed by the s/r machine to move from rack location x to y and c is a constant. The first term of Equation (1) corresponds to the travel time from the I/O point to the location L_{n-1} where the container will be stored. Note that we have implicitly assumed that the I/O location is at $(0,0)$. The second term refers to the travel time between locations L_{n-1} and L_{n+1}. A container is retrieved from L_{n+1}. The third term is the travel time back to the I/O location. The last term, c, is the constant term for pick-up and deposit times, and perhaps the fixed time for the s/r machine to move from the load stand to the rack at location $(0,0)$. The constant c might also be used to adjust for non-instantaneous acceleration and deceleration.

We will we assume that

$$||x - y|| = |x_1 - y_1| \vee |x_2 - y_2| \tag{2}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $\vee$ denotes "maximum of."

These model assumptions will be explored in more detail in the following subsections.

2.1 Travel Time Assumptions

Equation (2) implies two important assumptions: instantaneous acceleration and the Chebyshev metric (L^{∞} norm).

The Chebyshev travel metric arises from the fact that the s/r machine travels in the horizontal and vertical directions simultaneously. While one motor powers the mast horizontally back and forth along the guide rails, another motor advances the shuttle table vertically up and down the mast. Consequently, the time to travel between any two points in the rack can be expressed as the maximum of the time required to traverse the horizontal distance between the two points, and the time required to traverse the vertical distance between the two points. This travel metric is commonly referred to as the Chebyshev metric. If the time required to traverse the entire length of the rack from front to back is equal to the time required to traverse the entire height of the rack from bottom to top, the rack is said to be square-in-time (SIT). Note that the motors may have different speeds, so the rack may be square-in-time but not square physically.

A normalized square-in-time rack is depicted in Figure 2. Note that all the points on the two line segments joining $(0, r)$ with (r, r) and (r, r) with $(r, 0)$ are exactly r units of time from the origin where $0 \leq r \leq 1$.

Instantaneous acceleration means that when the s/r machine begins to move, it instantaneously accelerates to its maximum travel speed. Deceleration is also instantaneous. This is not the case in practice.

A more accurate model would still have the property, discussed in the next paragraph, that the length of time in the vertical and horizontal directions can be determined separately and the maximum of the two would be the travel time; however, the time in each direction is really a nonlinear function of the distance. This function would need to take into account several factors: the acceleration characteristics of the motors, the speeds of the motors (e.g., some motors have a regular speed and a "creep" speed), the braking characteristics, and, most importantly, the control policy. Some existing systems are not even monotonic—it may take more time to get to a closer position than a position further away—since the control policy may use the creep speed for the entire trip to the closer position. In spite of these problems, we have opted for the simpler model which seems to capture the basic behavior of the system; future models may refine this aspect of the model.

Equation (2) and the assumption that the lower left hand corner of the rack is at $(0, 0)$ and the upper right hand corner at $(1, b)$ imply that the rack has been normalized to account for the speed of the motors and the size of the rack. Note that b, also called the *shape factor*, is usually associated with the height of the rack. Typical shape factors range from .7 to 1.0. Also note that for SIT racks, b= 1.0. Other quantities, such as pick times, are normalized in a similar fashion.

2.2 Single Commands and Dual Commands

During a dual command (DC) cycle, both a storage and retrieval are performed. The s/r machine picks up a container from one of the pick positions, travels loaded to the dedicated storage location of that container, deposits the container in that location, travels empty to the next retrieval location, picks up the container from the location, travels loaded back to the I/O point, and deposits the container on the empty pick position. In other words, a DC cycle consists of two loaded travel legs from and to the I/O point, an empty travel between legs, two pick-ups and two deposits. In our analysis, we assume that there is an infinitely long list of retrieval requests. Hence, every cycle is a DC cycle. In some applications, it would be more appropriate to assume single command cycles, i.e., the s/r machine visits only one rack location before returning to the I/O point. For example, if the s/r machine placed the retrieved containers

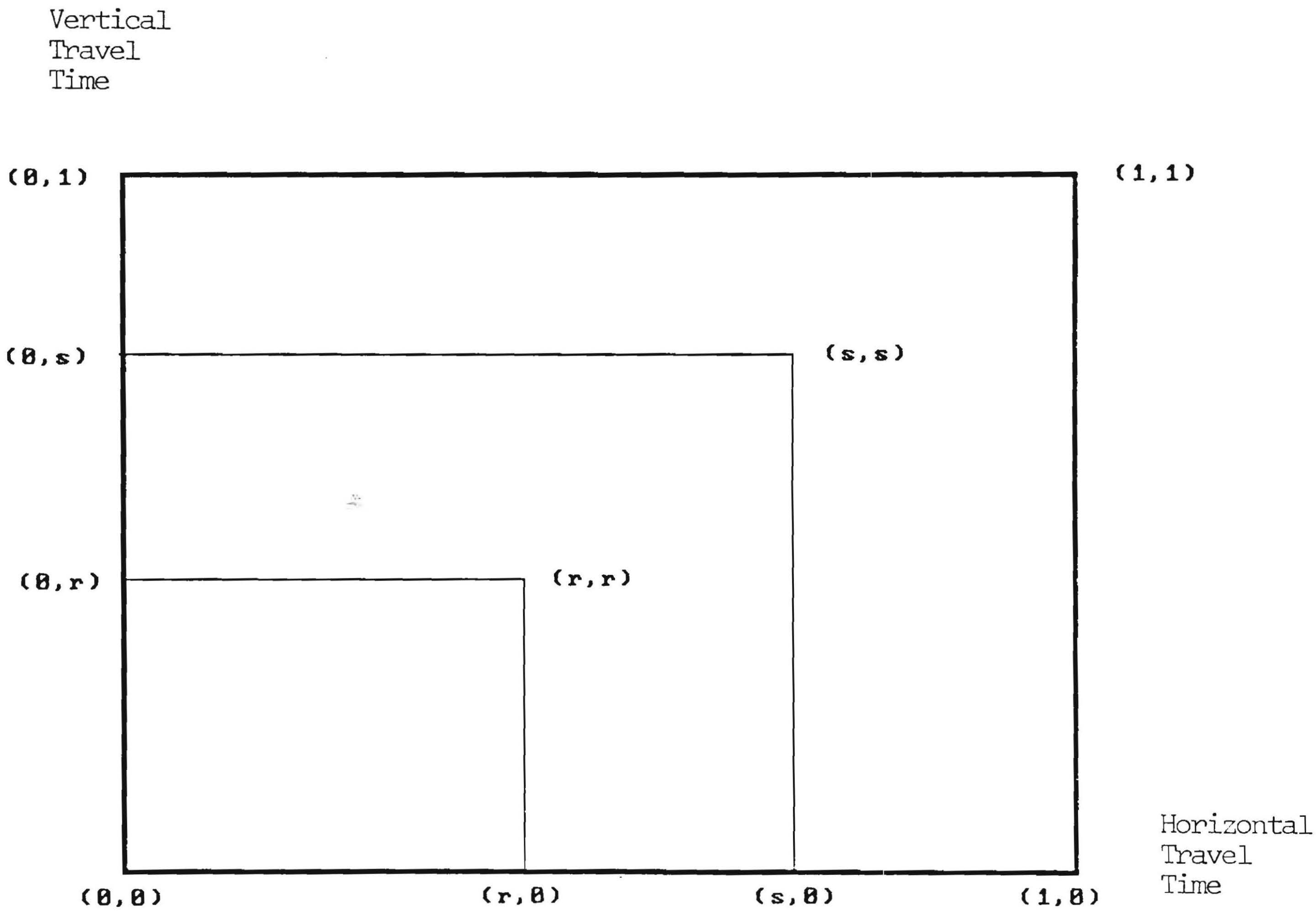

Figure 2: A Normalized SIT Rack.

on a conveyor, the s/r machine might perform several retrievals (stores) before storing (retrieving) any containers. The distribution of the travel time for single command (SC) cycles has been obtained by most authors who have obtained results on the expected travel time with either single or dual command cycles [6,11,13]. For the busy miniload system with a picker and two pick positions, the only reasonable assumption is that the storage and retrievals are interleaved; hence, dual command cycles.

2.3 Retrieval Sequencing and Storage Location Assignment

Our analysis assumes that retrievals are processed on a first-come-first-served (FCFS) basis. Hence, we assume that the sequence of random locations are independent; see Corollary 1. (Note, we are overlooking the fact that dependence may be induced by certain containers tending to be requested successively; for example, suppose parts from container j are mated with parts from container i. Whenever, container i is requested, container j tends to be requested next.) Most systems are operated in this FCFS fashion, though it has been pointed out that clever retrieval sequencing can decrease travel time [2,3,8,12] and increase system throughput [3]. We also assume that the random locations are uniformly distributed over the rack. This is the case when containers are assigned to rack locations without regard to their turnover, a common policy. Another alternative to decrease travel time is to assign the high turnover containers to the storage locations closest to the I/O point. Often, two or three classes of containers are distinguished based on their storage and retrieval activity. The high turnover containers are stored in the closest locations, the medium turnover in the middle locations, and the low turnover in the furthest locations from the I/O point. This is referred to as class-based storage. The limiting case, as the number of classes increases until each container is a class by itself, is referred to as full turnover-based storage. In this paper, we obtain closed-form results for uniform random locations over the rack.

2.4 Order Picking

The order picking activity for one container consists of a picker consulting a pick list to identify the items and quantities to select from the container, searching for the items in the container, and extracting the items from the container. This entire activity will be referred to as processing a container or simply the pick time. The time required to process the nth container in a sequence is a random variable and is denoted by P_n. We will give explicit results for two pick-time distributions: exponentially distributed pick times and deterministic pick times. The exponential distribution was chosen based on data from an existing facility. The deterministic case might be appropriate for robotic picking.

Other distributions, even empirical distributions based on work sampling, can be incorporated.

3 Literature Review

Previous research in AS/RS design and performance analysis includes simulation and analytical modeling [1,6,11,13], cost model development [1,4,18], and improvement strategies [3,8,9,12,16]. An exhaustive review of the literature in each of these areas is beyond the scope of this paper. The interested reader should see [1,3] for such reviews. However, since the focus of this paper is analytical modeling, we review the published literature concerning analytical models of automated storage and retrieval systems below.

Several authors have published results concerning analytical derivations of measures of performance for AS/RS travel time. In a little known paper, Gudehus [11] derived expressions for the expected value of single command and dual command travel time as a functions of s/r machine travel speed including the effects of acceleration [11]. Hausman, Schwarz, and Graves [13] developed an expression for expected single command travel time for SIT racks and class-based storage. Graves, Hausman, and Schwarz [10] later developed an expression for expected dual command travel time for SIT racks and class-based storage. Bozer and White [6] developed expressions for expected single and dual command cycles for racks of general shape, random storage locations, and a variety of I/O locations. Recently, Han, McGinnis, Shieh, and White [12] derived an analytical expression for expected dual command travel time for a general rack shape, random storage locations, and nearest neighbor retrieval sequencing.

The first work addressing the interaction of travel time and pick time in end-of-aisle order picking from an AS/RS was published in 1986 [3]. That work develops expected transaction time expressions for exponential and deterministic pick times under the assumption that s/r travel time is uniformly distributed with the same mean as the true distribution and a variance which has been empirically determined to be near the true variance.

In summary, previous work in analytical modeling yielded expressions for expected travel time expressions for a variety of assumptions concerning assignment and operating policies, I/O point location, and rack shape. In addition, the distribution of single command cycles has been determined. The distribution of dual command cycles has not. In that light, the major contributions of this work are the derivation of the probability distribution of dual command s/r machine travel time and the incorporation of that distribution in the derivation of exact expressions for the system throughput.

4 Analysis of Throughput

Let C_n denote the total travel time needed by the s/r machine during the nth dual command cycle, $T_0 = 0$, and

$$T_{n+1} = T_n + (C_{n+1} \vee P_{n+1}) \text{ for } n = 0, 1, \ldots.$$

Then T_n represents the time at which the s/r machine begins to move the nth container from a pick position and returns it to the rack; T_n also represents the time at which the order picker begins to select items from the (n + 1)st container. Let $N(t)$ denote the number of containers processed during $[0, t]$. More precisely, we define $N(t)$ as $N(t) \geq n \Leftrightarrow T_n \leq t$. We assume that $N(t) < \infty$, a.s., but $N(t) \rightarrow \infty$, a.s., as $t \rightarrow \infty$. System throughput will be defined as the long run average number of containers processed per hour, i.e.,

$$\lim_{t\to\infty} \frac{N(t)}{t}$$

provided this limit exists. The next theorem gives conditions under which this limit exists and an expression for this limit.

Theorem 1 *Let $D_n = (L_n, P_n)$. If $\{D_0, D_1, \ldots\}$ is stationary and ergodic, then m^{-1} is the system throughput where*

$$m = E[C_n \vee P_n] \tag{3}$$

In addition, the long run proportion of time that the s/r machine is busy is $E[C_n]/m$ and for the picker $E[P_n]/m$.

Remark 1 *This is similar to results from renewal theory. However, $N(t)$ is not a renewal process even if the D_n's are i.i.d. since the container retrieved from location L_n in cycle $n - 1$ is stored in location L_n in cycle $n + 1$. This creates dependencies among the cycles.*

Proof Throughout the proof, we will suppress the a.s.. First, we show that $T_n/n \rightarrow m$. The process (C_n, P_n) is also stationary and ergodic as can be seen from Propositions 6.6, 6.4 and 6.31 of [7]. From Corollary 6.23 of [7]

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{C_k \vee P_k}{n} = E[C_1 \vee P_1] = m.$$

Now, we use sample path arguments to complete the proof. Clearly,

$$T_{N(t)} \leq t \leq T_{N(t)+1}.$$

Dividing through by N(t), gives

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}.$$

Letting t go to infinity and noting that $\frac{N(t)+1}{N(t)}$ converges to 1, yields

$$m \leq \lim_{t\to\infty} \frac{t}{N(t)} \leq m$$

which shows that the throughput is m^{-1}.

The proportion of time that the s/r machine is busy in the first n transactions is $(C_1 + \cdots + C_n)/T_n$. Dividing the numerator and denominator by n and letting n go to infinity gives the desired result. The proportion of time that the picker is busy can be proven analogously.

□

In order to compute system throughput, we need the joint distribution of C_n and P_n. However, C_n and P_n will be independent in the following situation.

Corollary 1 *If $D_0, D_1, \ldots$ are independent and identically distributed, then C_n and P_n are independent and*

$$m = \int_0^\infty (1 - \Pr\{C_n \leq t\} \Pr\{P_n \leq t\})dt \tag{4}$$

Remark 2 *Note that we do* not *need L_n and P_n to be independent which is fortunate since under class based storage they are often dependent.*

Proof The independence of C_n and P_n follows from noting that C_n is only a function of L_{n-1} and L_{n+1} (and not L_n). Equation (4) follows directly from (3) after using the independence and the fact that the expected value of a non-negative random variable is the integral of the complementary distribution function.

□

At this point, we need the distribution of C_n which is derived in the following section.

5 The Distribution Function of Dual Command Travel Time

In general, there does not appear to be a closed form expression for the distribution of the dual command cycle. In order to derive an expression which is analytically tractable, we will analyze the continuous as opposed to the discrete rack. That is, we assume that the s/r machine L_n can be any point on the unit square. This can be considered as an approximation to the discrete rack when the number of rack openings is large and each opening has a small probability of being visited. Some empirical work indicates that there is little difference between the discrete and continuous case. In addition, limit theorems show that under certain assumptions the behavior of the cycle time in the discrete rack converges to the behavior of the continuous rack as the number of cells increases. We will also assume that the locations L_0, L_1,... are uniformly distributed over the rack. This corresponds to actual systems in which the containers are stored in the racks without regard to the frequency that they will be requested. Lastly, we will assume that the rack is square-in-time, and that the rack can be represented as a continuum of storage locations. This continuous approximation of the discrete rack has been empirically validated in a number of studies [6,13,15]. Under the above assumptions, we can derive the distribution of the dual command cycle.

Let $F(\cdot)$ be the cumulative distribution function of $C_n - c$, the variable portion of the travel time from Equation (1).

Theorem 2 *If L_{n-1} and L_{n+1} are independent, uniformly distributed random points in the unit square, then*

$$F(x) = \begin{cases} \frac{25}{576}x^4 & \text{if } 0 \leq x \leq 2 \\ \frac{1}{36}x^4 - \frac{3}{2}x^2 + 6x - \frac{23}{4} & \text{if } 2 < x \leq 3. \end{cases} \tag{5}$$

Remark 3 *A graph of the probability density function appears in Figure 3. The expected value of $C_n - c$ as computed from (5) is 9/5 which agrees with the known results in [6,10,11]. The variance is 53/300.*

Remark 4 *Equation (5) gives the distribution function for the variable portion of the travel time. In Equation (4), we need the distribution of C_n which is*

$$\Pr\{C_n \leq t\} = F(t - c)$$

Proof We wish to generate two independent, uniformly distributed random points in the unit square. The simplest way would be to assume that each of the coordinates are independent, uniform [0,1] random variables. However, it will be more convenient to first assume that the point falls on some curve, and then given that the point falls on that curve, its location on the curve will be uniformly distributed. We will select the curves $\{x \in [0,1]^2 : ||x|| = r\}$, for $0 \leq r \leq 1$. We will call the curve generated by a fixed value of r, curve r[1]. Curve r corresponds to the points on the two line segments joining $(0, r)$ with (r, r) and (r, r) with $(r, 0)$ in Figure 2.

We select these curves for three reasons. First, they partition the unit square. Second, if the point falls on curve r, the point will be r units from the origin. Third, even under class based storage with L-shaped regions or full turnover

[1] The approach here is in the spirit of the approach used in the appendix to [10].

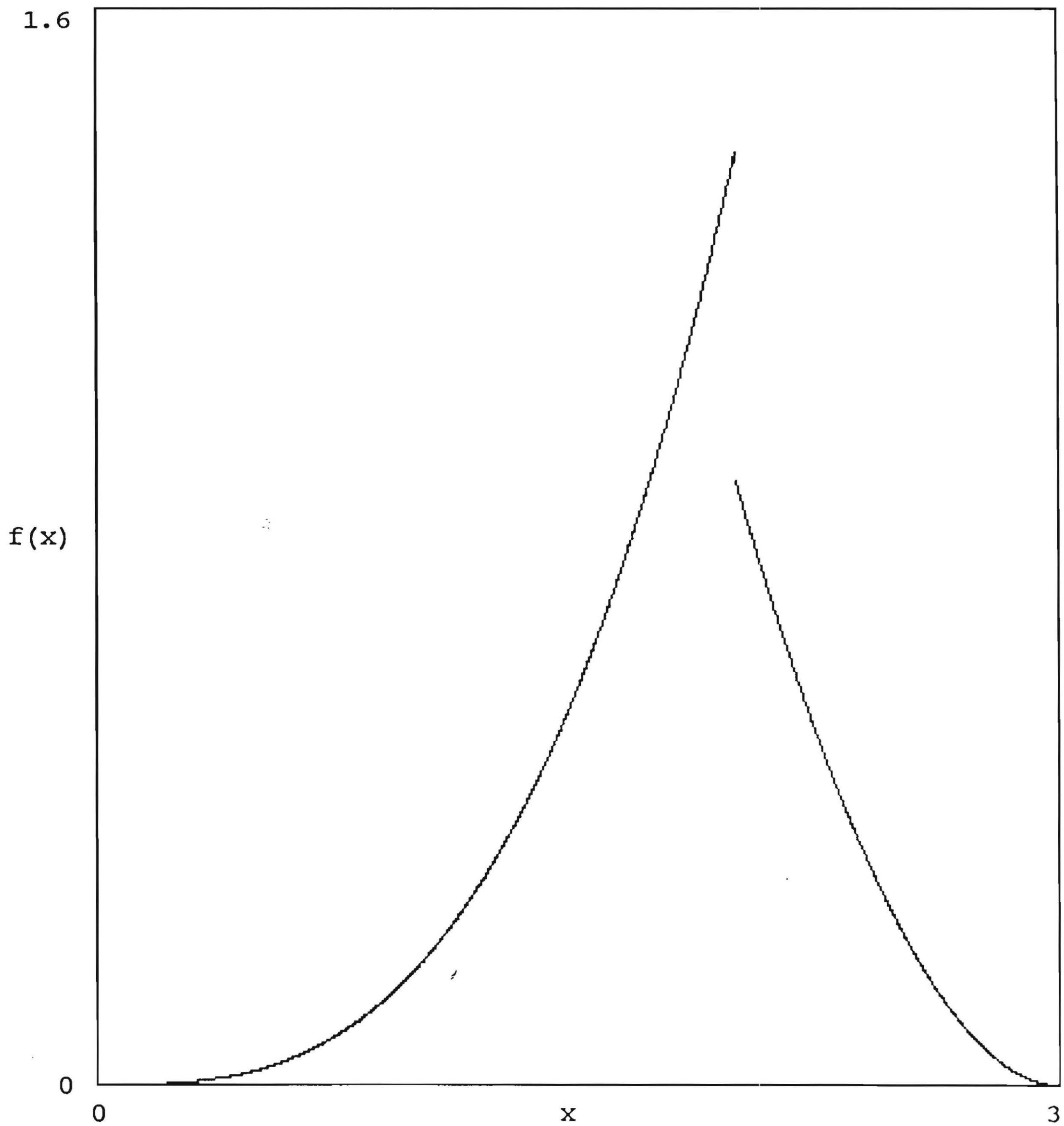

Figure 3: The probability density function of Dual Command Travel Time

based storage, the point will still be uniformly distributed over curve r given that it is on curve r. (For class based storage with I- shaped regions, it might be more convenient to select the vertical line segments as opposed to the L-shaped curves in order to save the third property, even though the second property will be lost.)

The probability density function for the point landing on curve r in the uniform case is $g(r) = 2r$ for $0 < r < 1$. Since the two points, r and s, are independent, the joint probability density function of one point on curve r and the other on s is $g(r)g(s)$. If we let R be the shorter and S the longer leg, i.e., $R = \|L_{n-1}\| \wedge \|L_{n+1}\|$, $S = \|L_{n-1}\| \vee \|L_{n+1}\|$, and

$$F_{r,s}(t) = \Pr\{ \|L_{n-1} - L_{n+1}\| \leq t | R = r, S = s\}, \tag{6}$$

then we have

$$F(x) = \int_0^1 \int_0^s F_{r,s}(x - r - s)\, 2!\, g(r)g(s)\, dr\, ds. \tag{7}$$

By symmetry, we can assume that the point on curve s falls on the vertical portion of curve s and that we have two equally likely cases depending on whether the point on curve r falls on the horizontal or vertical portion. Hence,

$$\begin{aligned} F_{r,s}(t) &= \frac{1}{2}\Pr\{\|r(U_1, 1) - s(1, U_2)\| \leq t\} \\ &+ \frac{1}{2}\Pr\{\|r(1, U_1) - s(1, U_2)\| \leq t\} \end{aligned}$$

where U_1 and U_2 are independent, uniform $[0, 1]$ random variables. After some algebra this reduces to

$$F_{r,s}(t) = \begin{cases} F_A(t) & \text{if } t \geq s \\ F_B(t) & \text{if } t > s - r, r < t < s \\ F_C(t) & \text{if } t > s - r, t \leq r,\ 2t \leq s \\ F_D(t) & \text{if } s > t > s - r, t \leq r,\ 2t > s \\ 0 & \text{if } t \leq s - r \end{cases} \tag{8}$$

where

$$\begin{aligned} F_A(t) &= 1 \\ F_B(t) &= \int_0^{(s-t)/r} \frac{ru + t}{2s} du + \frac{1}{2}\int_{(s-t)/r}^1 du + \frac{1}{2}(1 - \frac{s - t}{r}) \\ F_C(t) &= \int_0^{t/r} \frac{ru + t}{2s} du + \int_{t/r}^{(s-t)/r} \frac{t}{s} du + \int_{(s-t)/r}^1 \frac{s - (ru - t)}{2s} du \\ &+ \frac{1}{2}(1 - \frac{s - t}{r})(1 - \frac{r - t}{s}) \end{aligned}$$

$$
\begin{aligned}
F_D(t) &= \int_0^{(s-t)/r} \frac{ru+t}{2s} du + \frac{1}{2}\int_{(s-t)/r}^{t/r} du + \int_{t/r}^{1} \frac{s-(ru-t)}{2s} du \\
&+\frac{1}{2}(1-\frac{s-t}{r})(1-\frac{r-t}{s}).
\end{aligned}
$$

Equation (7) can be rewritten as follows.
For $0 < x < 2$,

$$
\begin{aligned}
F(x) &= \int_0^{\frac{x}{3}} \int_r^{\frac{x-r}{2}} F_A(x-r-s)2g(r)g(s)dsdr \\
&+ \int_{\frac{x}{3}}^{\frac{x}{2}} \int_{x-2s}^{\frac{x-s}{2}} F_B(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{x}{3}}^{\frac{2x}{5}} \int_{\frac{x-s}{2}}^{s} F_D(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{2x}{5}}^{\frac{x}{2}} \int_{\frac{x-s}{2}}^{x-\frac{3s}{2}} F_D(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{2x}{5}}^{\frac{x}{2}} \int_{x-\frac{3s}{2}}^{s} F_C(x-r-s)2g(r)g(s)drds.
\end{aligned}
$$

For $2 < x < 2.5$,

$$
\begin{aligned}
F(x) &= \int_0^{\frac{x}{3}} \int_0^{s} F_A(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{x}{3}}^{1} \int_0^{x-2s} F_A(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{x}{3}}^{1} \int_{x-2s}^{\frac{x-s}{2}} F_B(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{x}{3}}^{\frac{2x}{5}} \int_{\frac{x-s}{2}}^{s} F_D(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{2x}{5}}^{1} \int_{\frac{x-s}{2}}^{x-\frac{3s}{2}} F_D(x-r-s)2g(r)g(s)drds \\
&+ \int_{\frac{2x}{5}}^{1} \int_{x-\frac{3s}{2}}^{s} F_C(x-r-s)2g(r)g(s)drds.
\end{aligned}
$$

For $2.5 < x < 3$,

$$
F(x) = \int_0^{\frac{x}{3}} \int_0^{s} F_A(x-r-s)2g(r)g(s)drds
$$

$$+ \int_{\frac{x}{3}}^{1} \int_{0}^{x-2s} F_A(x-r-s)2g(r)g(s)drds$$

$$+ \int_{\frac{x}{3}}^{1} \int_{x-2s}^{\frac{x-s}{2}} F_B(x-r-s)2g(r)g(s)drds$$

$$+ \int_{\frac{x}{3}}^{1} \int_{\frac{x-s}{2}}^{s} F_D(x-r-s)2g(r)g(s)drds.$$

Using $g(r) = 2r$, the expressions for $F_A, \ldots, F_D$, and integrating yields (5).

□

6 Examples: Deterministic and Exponentially Distributed Pick Times

From Sections 4 and 5, we have the following result: If $D_n = (L_n, P_n)$ is independent and identically distributed and the pick locations, L_n, are uniformly distributed over a normalized, square-in-time rack then

$$m = \int_0^{\infty} (1 - F(t-c)\Pr\{P_n \leq t\})dt \tag{9}$$

where $F(x)$ is given in (5), c is the constant portion of the s/r travel time, and $\Pr\{P_n \leq t\}$ is the cumulative distribution function for the pick time. Hence, from Theorem 1, the system performance measures are:

- The system throughput is $1/m$.
- The picker utilization is $E[P_n]/m$ where $E[P_n]$ is the normalized expected pick time.
- The s/r machine utilization is $(1.8 + c)/m$.

To work out some examples, we will integrate Equation (9) for several pick time distributions to obtain closed form expressions for the critical number m.

Suppose the pick time is a constant d which might be reasonable under robotic picking. Thus,

$$\Pr\{P_n \leq t\} = \begin{cases} 1 & \text{if } t \geq d, \\ 0 & \text{if } t < d. \end{cases} \tag{10}$$

When $c \leq d < 2 + c$, Equation (9) simplifies to

$$\begin{aligned} m &= 9/5 + c - \frac{25}{576}cd^4 + \frac{25}{288}c^2d^3 - \frac{25}{288}c^3d^2 \\ &+ \frac{25}{576}c^4d - \frac{5}{576}c^5 + \frac{5}{576}d^5, \end{aligned} \tag{11}$$

and when $2 + c \leq d < 3 + c$,

$$\begin{aligned} m &= \frac{27}{5} + \frac{27}{4}c - \frac{23}{4}d - 6cd + \frac{3}{2}cd^2 - cd^4/36 - \frac{3}{2}c^2 d \\ &\quad + c^2 d^3/18 - c^3 d^2/18 + c^4 d/36 + 3c^2 + c^3/2 - c^5/180 \\ &\quad + 3d^2 - d^3/2 + d^5/180 \end{aligned} \tag{12}$$

For a second example, if the pick times are exponentially distributed with mean $1/\lambda$, then we would have

$$\Pr\{P_n \leq t\} = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{13}$$

For this distribution, Equation (9) simplifies to

$$\begin{aligned} m &= 9/5 + c - \frac{2/3}{e^{3\lambda + c\lambda}\lambda^5} - \frac{2}{e^{3\lambda + c\lambda}\lambda^4} - \frac{3/8}{e^{2\lambda + c\lambda}\lambda^5} \\ &\quad - \frac{3/4}{e^{2\lambda + c\lambda}\lambda^4} - \frac{15/4}{e^{2\lambda + c\lambda}\lambda^3} - \frac{1}{2e^{2\lambda + c\lambda}\lambda^2} + \frac{25/24}{e^{c\lambda}\lambda^5} \end{aligned} \tag{14}$$

7 Numerical Example

Consider a rack 30 feet high and 120 feet long serviced by an s/r machine which travels 90 feet per minute vertically and 360 feet per minute horizontally. Hence, the s/r machine takes 1/3 minute to travel from the front to the back and also from the bottom to the top of the rack. The expected processing time per container is 1.5 minutes, and the constant time for each pick-up or deposit is 0.15 minutes.

Since we have assumed that the s/r machine takes one unit of time to travel from the front to the rear of the back, it is easier to normalize the data to take this into account. Since the s/r machine takes 1/3 minute to travel from the front to the back, we assume that the units of time are 1/3 minutes. Converting the data to these time units results in the rack being one time unit long, one time unit high, the expected processing time being 4.5 time units, and each pickup or deposit requiring .45 time units. Since there are two pickups and two deposits per dual command cycle, c is $4 * .45 = 1.8$ time units. Henceforth, all system parameters will be given in normalized time units unless otherwise specified.

The expected dual command travel time can be calculated from Equation (7); however, it has already been determined to be $1.8 + c$ in [6,13] which in our example is 3.6 time units.

We have not completely specified the pick time distribution, only its mean. Below, we calculate the throughput for two particular cases: deterministic and exponentially distributed pick times.

	Pick Times (mean of 1.5 minutes)	
	Deterministic	Exponentially Distributed
Picker Capacity	40/hour	
S/R Machine Throughput	50/hour	
Picker Utilization	99.9%	79.9%
S/R Machine Utilization	79.9%	63.9%
System Throughput	40/hour	32/hour

Table 1: Results for the Numerical Example

For deterministic pick times with $2+c < d < 3+c$, the expected transaction time m is calculated from Equation (12). With $c = 1.8$ time units and $d = 4.5$ time units, we have $m \approx 4.50064$ time units or 1.50021 minutes. Hence, the throughput is $1/m \approx 0.6666$ containers per minute or roughly 39.99 containers per hour. The picker utilization is $E[P_n]/m \approx 4.5/4.50064 \approx .9998$ or 99.9%. Similarly, the s/r machine utilization is $E[C_n]/m \approx 3.6/4.50064 \approx 0.7999$ or 79.9%.

For exponentially distributed pick times, the expected cycle time is calculated from Equation (14). With $c = 1.8$ time units and $1/\lambda = 4.5$ time units or equivalently $\lambda = 2/9$, we have $m \approx 5.631$ time units or $m \approx 1.877$ minutes. Hence, the throughput is $1/m \approx 0.533$ containers per minute or roughly 32 containers per hour. The picker utilization is $E[P_n]/m \approx 4.5/5.631 \approx 0.799$ or 79.9%. The s/r machine utilization is $E[C_n]/m \approx 3.6/5.631 \approx 0.639$ or 63.9%.

Note, that even though the mean pick time is the same for both cases, the variability of the exponential distribution has decreased the picker utilization, s/r machine utilization, and system throughput.

Also, note that if we analyzed the s/r machine in isolation, we would conclude that the s/r machine can process 50 containers/hour; similarly, the picker can process 40/hour. However, it would be hasty to state that the picker is the bottleneck and that the system throughput is 40 containers/hour. In the exponentially distributed case, the throughput is only 32/hour and both the picker and s/r machine are affecting the throughput. Decreasing either the pick time or the s/r machine travel time would increase throughput.

8 Conclusions

The miniload automated storage/retrieval system is frequently employed in spare parts order picking, kitting for assembly operations, file storage and retrieval, and small parts distribution. We have, under certain assumptions, derived exact expressions for the maximum throughput of a miniload AS/RS in Theorem 1 and Corollary 1. To use these expressions, we needed the distribution of the dual command travel time which we derived under certain assumptions in Theorem 2. The distribution of dual command travel time should be of independent interest. Combining these results yields simple analytical expressions for the throughput of a miniload AS/RS as shown in Section 6. These final results should be useful to designers of miniload AS/R systems. In fact, the final results are simple enough to program into a pocket calculator which we have used to compute the minimum number of aisles given the pick time distribution and the required system throughput and storage capacity.

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Throughput Expressions and Bounds for End-of-Aisle Order Picking with Activity Based Storage *

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Abstract

In a previous paper, we obtained expressions for the throughput of square-in-time miniload systems with uniformly distributed bin retrieval activity and known pick time distribution. In this paper, we extend these results in two directions. First, we consider systems with two storage classes: high and low activity. Second, even if only partial information about the pick times is available, such as the mean pick time or that all pick times lie in the interval $[u, v]$, we find tight upper and lower bounds on the system throughput. We compute these bounds for several types of limited information.

Keywords: Miniload, Order Picking, Throughput, Activity Based Storage, Throughput Bounds, Class Based Storage, Convex Ordering, Stochastic Ordering, Automated Storage/Retrieval System

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1 Introduction

Miniload automated storage and retrieval systems are in the class of end-of-aisle or part-to-picker order picking systems [6]. Miniload systems are popular for small parts order picking because they provide excellent space utilization, excellent item security, and accurate item picking [5]. In fact, over 500 miniload systems are currently in use in the U.S. [11]. However, at over $300,000 per aisle [17] and with limited ability to reconfigure, the initial system design must be accurate. In addition, system designers must make every effort to maximize system productivity in order to increase the return on a significant investment. System productivity can often be enhanced by taking advantage of the fact that a small minority of the items housed in the system generate a large majority of the bin retrieals. We show how this activity skew effects system throughput.

Some analytical expressions for the throughput of a miniload AS/RS are available in [4]. However, to compute the throughput, the distribution of the pick time is needed. Frequently, the pick time distribution is unknown, but some partial information about the distribution such as its mean or its mean and range may be known. In this paper, we determine tight upper and lower bounds on throughput for several different types of partial information.

2 Physical Description of a Miniload AS/RS

A typical miniload system (Figure 1) is comprised of multiple aisles of storage racks, a storage/retrieval (s/r) machine operating in each aisle, numerous modular storage bins for housing the items, and two load stands (or pick positions) at the end of each aisle to facilitate order picking. The load stands are arranged such that each aisle has a left and a right pick position. While the order picker is extracting items from the bin in one pick position, the s/r machine returns the container in the other pick position to its permanent, dedicated location in the rack, and returns with the next bin to be processed. If the picker finishes before the s/r machine returns with the next bin, the picker waits idle. If the s/r machine returns with the next bin before the picker is finished, the s/r machine waits idle.

The *dual command cycle time* is the length of time the s/r machine takes to pick up a bin from the pick station, return the bin to its home location, travel empty to the next bin location, retrieve the bin, travel to the I/O point, and deposit the bin on the empty table. The *pick time* is the length of time needed by the picker to process the bin. The processing activities may include extracting the items, documenting the tasks, counting and weighing items, interruptions, and restocking.

Not all miniload systems are configured like the one shown in Figure 1. Systems may be configured with more than two pick positions per aisle, with a conveyor delivery system to deliver containers to "remote" order pickers, or

Figure 1: Miniload AS/RS, reprinted from [17].

with multiple input/output (i/o) points per aisle. However, it is the typical configuration, with one picker, two pick positions per aisle, and one i/o point located at the end of each aisle, that is modeled here. Since each aisle operates independently, we limit our analysis to a single aisle.

3 Model Assumptions

In our analysis we assume that retrievals are processed on a first-come-first-served (FCFS) basis and that there is an infinite queue of requests. Thus, we are determining the maximum possible throughput; i.e. the throughput capacity. We also assume the sequence of requested bin locations are independent and identically distributed. Therefore we are not allowing dependence induced by the fact that certain bins tend to be requested successively. For example, items in bin 2 may be associated with items in bin 8. Then, whenever, bin 2 is requested, bin 8 tends to be requested next [6].

We make three important assumptions concerning the characteristics of s/r machine travel within the aisle of storage racks. First, we assume that the travel times are calculated with the Chebychev metric. Since one motor powers the mast horizontally back and forth along the guide rails, and another motor advances the shuttle table vertically up and down the mast, the assumption seems valid. Our second assumption is that the s/r machine accelerates and decelerates instantaneously to/from its maximum travel speed. This is not the case in practice, however, new s/r machines are approaching instantaneous acceleration/deceleration. Finally, we assume that the rack is square-in-time. A square-in-time rack is one in which the time required for the s/r machine to travel horizontally from the front of the rack to the back is equal to the time required to travel vertically from the bottom of the rack to the top. Since the horizontal travel speed is typically 2 to 4 times the vertical speed, the rack may be square-in-time but not square physically.

We assume that the rack is divided into two regions for bin location assignments (Figure 2). The region closest to the I/O point, region I, is assumed to house the high turnover items. Region I is defined on the normalized rack by the points $(0,0),(a,0),(a,a)$, and $(0,a)$. Region II occupies that portion of the rack not contained in region I. Within each region, we assume that the random sequence of location visits are uniformly distributed. This is the case when items are classified as "fast" or "slow" movers and items are assigned to locations within a region without additional consideration of their turnover.

For a more detailed discussion of these assumptions, see [4].

Figure 2: Square-in-Time Rack with Two Activity Zones

4 Literature

Several authors have conducted research on AS/RS travel time models. Gudehus [8] derived expressions for the expected single command travel time in an AS/RS as a function of the system's length, height, and horizontal and vertical travel speeds and acceleration. Hausman, Schwarz, and Graves [10] developed expressions for expected single command travel in square-in-time (SIT) racks with class-based dedicated storage. Graves, Hausman, and Schwarz [7] later developed an expression for dual command travel in SIT racks under class-based storage. Bozer and White [3] developed expressions for expected single and dual command travel time for racks of general shape in a variety of configurations. Finally, Han, McGinnis, Sheih, and White [9] derived an analytical expression for expected dual command travel time for a general rack shape, random storage, and nearest neighbor retrieval sequencing.

To our knowledge, the only published works which address the interaction of travel and processing time is that of Bozer and White [2] and Foley and Frazelle [4]. Bozer and White [2] developed approximate expressions for the expected rate at which bins are processed in a miniload AS/RS when bin processing times are assumed to be exponentially or uniformly distributed. They assume dual command travel time is uniformly distributed with the same mean as the true distribution and with a variance which has been empirically determined to be close to the true variance. Foley and Frazelle [4] derived an exact expression for expected order picking productivity in a square-in-time miniload AS/RS for general distributions of bin processing times. In both cases, a random storage policy was assumed.

5 Convex Ordering and Throughput Bounds

5.1 Approach

Our approach to determining upper and lower bounds on the throughput is as follows. Partial information on the pick time distribution, e.g. the mean and variance, defines a set of possible pick time distributions. If there are extremal distributions in this set, i.e., if there is a best and worst possible pick time distribution no matter what rack type, then we can use these extremal distributions to bound the throughput.

We consider the following sets of distribution functions corresponding to different types of partial information (the parenthetical remarks give the corresponding partial information)

$\mathcal{G}^+$: all distribution funtions of nonnegative random variables (no information),

$\mathcal{G}^+_\mu$: all distribution functions in $\mathcal{G}^+$ with mean μ (known mean),

$\mathcal{G}^+_{\mu,\sigma^2}$: all distribution functions in $\mathcal{G}^+_\mu$ with variance σ^2 (known mean and variance),

$\mathcal{G}^{[u,v]}_\mu$: all distribution functions G in $\mathcal{G}^+_\mu$ with $G(u-) = 0$, $G(v+) = 1$, and $0 \leq u \leq \mu \leq v$ (known mean and range),

$\mathcal{G}^{[u,v]}_{\mu,\sigma^2}$: all distribution functions in $\mathcal{G}^{[u,v]}_\mu$ with variance σ^2 (known mean, variance, and range),

$\mathcal{G}^{NBUE}_\mu$: all distribution functions in $\mathcal{G}^+_\mu$ which are NBUE (known mean and NBUE).

The distribution function of a random variable X is NBUE (New Better than Used in Expectation) if $E[(X-t) \mid X > t] \leq E[X]$ for all $t > 0$. Roughly, NBUE means that the expected remaining pick time from a partially processed tray is smaller than the mean pick time from a new tray. For convenience, we will often shorten phrases like "...the distribution function of the random variable X is an element of $\mathcal{G}^{NBUE}_\mu$" to " ...X is in $\mathcal{G}^{NBUE}_\mu$."

5.2 Convex Ordering and System Throughput

In this subsection, we show that system throughput decreases with a "larger" pick time distribution. Of course, there are numerous ways to say that one distribution function is larger than another. We will use the convex ordering described in [12,15] among others. A random variable X is convexly larger than a random variable Y if $E[f(X)] \geq E[f(Y)]$ for all nondecreasing convex functions f. Similarly, a distribution function F is convexly smaller than a distribution function G, if a random variable with distribution F is convexly smaller than a random variable with distribution G.

Let C denote a dual command travel time. Let P_i denote a pick times with distributions $G_i, i = 1, 2$ respectively. Throughout this paper, we will assume that P_i and C are independent for $i = 1, 2$, and that the system throughput is $1/E[P \vee C]$. See Theorem 1 and Corollary 1 for sufficient conditions. By the way, the independence between the dual command travel time and the p can hold even if the pick time from a bin is correlated with the bin's location; see [4].)

The next result states that the throughput decreases if the pick time increases convexly.

Proposition 1 *Assume that $P_i andC$ are independent for $i = 1, 2$. If $P_1 \leq_c P_2$, then $1/E[P_1 \vee C] \geq 1/E[P_2 \vee C]$;.*

Proof Let $g(t) = E[P_1 \vee C \mid P_1 = t]$. By independence, $g(t) = E[t \vee C]$ which is easily shown to be a nondecreasing, convex function of t. Hence,

$$E[P_1 \vee C] \quad = \quad E[E[P_1 \vee C \mid P_1]]$$

$$
\begin{aligned}
&= E[g(P_1)] \\
&\leq E[g(P_2)] \\
&= E[P_2 \vee C]
\end{aligned}
$$

□

Repeated application of Proposition 1 yields the following corollary which states that convexly increasing the pick time, the dual command travel time, or both decreases throughput.

Corollary 1 *Assume that P_i and C_j are independent for $i = 1, 2$ and $j = 1, 2$. If $P_1 \leq_c P_2$ and $C_1 \leq_c C_2$, then*

$$
\begin{aligned}
E[P_1 \vee C_1] \leq E[P_1 \vee C_2] \leq E[P_2 \vee C_2] \\
E[P_1 \vee C_1] \leq E[P_2 \vee C_1] \leq E[P_2 \vee C_2]
\end{aligned}
$$

5.3 System Performance Bounds

In this subsection we determine bounds on the throughput corresponding to some of the various types of partial information available. In many cases, the bound is determined by a certain pick time distribution being the best or worst possible. For example, if the pick times have a known mean μ and are NBUE, the best system has deterministic pick times and the worst has exponentially distributed pick times. It will be useful to define a few random variables with distributions which turn out to be extremal for certain types of partial information. Let

- $X_\mu^{u,v}$ be a random variable which takes either the value u or v and has mean μ;
- X_μ^+ be a random variable with distribution function

$$
G_\mu^+(x) \equiv \begin{cases} 0 & \text{for } x < 0 \\ \sigma^2/(\mu^2 + \sigma^2) & \text{for } 0 \leq x < (\mu^2 + \sigma^2)/(2\mu) \\ S_{\mu,\sigma^2}(x) & \text{for } (\mu^2 + \sigma^2)/(2\mu) \leq x \end{cases}
$$

where

$$
S_{\mu,\sigma^2}(x) \equiv 1 + (x - \mu)/\sqrt{(x - \mu)^2 + \sigma^2};
$$

- $X_{\mu,\sigma^2}^{0,v}$ be a random variable with distribution function

$$
G_{\mu,\sigma^2}^{u,v}(x) \equiv \begin{cases} 0 & \text{for } x < 0 \\ \sigma^2/(\mu^2 + \sigma^2) & \text{for } 0 \leq x < (\mu^2 + \sigma^2)/(2\mu) \\ S_{\mu,\sigma^2}(x) & \text{for } (\mu^2 + \sigma^2)/(2\mu) \leq x < x(v) \\ S_{\mu,\sigma^2}(x(v)) & \text{for } x(v) \leq x < v \\ 1 & \text{for } v \leq x \end{cases}
$$

where

$$\begin{aligned} x(v) &\equiv (\mu^2+\sigma^2-u(v)^2)/[2(\mu-u(v))] \\ u(v) &\equiv [m((m-v)^2+\sigma^2)-\sigma^2 v]/(m-v)^2; \end{aligned}$$

- $X^{u,v}_{\mu,\sigma^2}$ be a random variable with the same distribution as the random variable $u + X^{0,v-u}_{\mu-u,\sigma^2}$;
- X^{NBUE}_{μ} be an exponential random variable with mean μ.

Theorem 1 *Let C denote the dual command travel time and P the pick time. If the throughput is $TPT \equiv 1/E[P \vee C]$, then the following bounds hold and are sharp:*

a) $0 \le TPT \le 1/E[C]$;

b) *if* $P \in \mathcal{G}^{+}_{\mu}$, *then* $1/(\mu + E[C]) \le TPT \le 1/E[\mu \vee C]$;

c) *if* $P \in \mathcal{G}^{+}_{\mu,\sigma^2}$, *then* $1/E[X^{+}_{\mu,\sigma^2} \vee C] \le TPT \le 1/E[\mu \vee C]$.

d) *if* $P \in \mathcal{G}^{[u,v]}_{\mu}$, *then* $1/E[X^{u,v}_{\mu} \vee C] \le TPT \le 1/E[\mu \vee C]$;

e) *if* $P \in \mathcal{G}^{[u,v]}_{\mu,\sigma^2}$, *then* $1/E[X^{u,v}_{\mu,\sigma^2} \vee C] \le TPT \le 1/E[\mu \vee C]$;

f) *if* $P \in \mathcal{G}^{NBUE}_{\mu}$, *then* $1/E[X^{NBUE}_{\mu} \vee C] \le TPT \le 1/E[\mu \vee C]$.

Remark 1 *The crude bound in a) is the best possible if no information about the service times are available. If the mean pick time is known, the system with maximum throughput in all cases turns out to be the system with deterministic pick times. Lower bounds are often more useful in practice. The easiest lower bound to use is in b). In f) and d), the extremal pick time distribution turn out to be rather simple: an exponential distribution in f) and a two-point distribution in d).*

Remark 2 *In [4], closed form expressions were given for the throughput of a square-in-time, continuous, uniform rack with deterministic pick times and with exponential pick times. Using Theorem 1, these expressions give all of the bounds except for the lower bounds c), d), and e).*

Remark 3 *The throughput bounds in Theorem 1 also give bounds on the utilization of the picker and the s/r machine since the picker utilization is simply $E[P]/E[P \vee C] = E[P]\ TPT$ and the s/r machine utilization is simply $E[C]/E[P \vee C] = E[C]\ TPT$[4].*

Proof We need to show that these bounds hold and are sharp. For the lower bounds in d) and f) and all of the upper bounds on TPT, the results follow from Section 1.9 of [15] on extremal elements of sets of distribution functions. For the lower bound in c), use Theorem 3 of [16]. For the lower bound in e) when $u = 0$, also use Theorem 3 of [16]. To extend this to $u > 0$, argue that if X is an extremal element of $\mathcal{G}^{[0,v]}_{\mu,\sigma^2}$ then $X + u$ is an extremal element of $\mathcal{G}^{[u,v+u]}_{\mu+u,\sigma^2}$. To show that the lower bound in a) is sharp, consider a sequence of systems where the mean pick time in the nth system is n (implying the throughput is no greater than $1/n$). To obtain the lower bound in b), note that $E[P \vee C] \leq (\mu + E[C])$. To show that this bound is sharp note that

$$
\begin{aligned}
E[X^{0,v}_{\mu} \vee C] &\geq \frac{\mu}{v}v + (1 - \frac{\mu}{v})E[C] \\
&\rightarrow \mu + E[C] \text{ (as } v \rightarrow \infty).
\end{aligned}
$$

□

6 Distribution of DCT with 2 Activity Classes

In an earlier paper [4], we derived the distribution of the dual command travel time for a square-in-time, continuous rack without activity based storage. In this section we extend these results to allow activity based storage with two classes: high and low turnover. We assume a continuous, square-in-time rack.

Let $F(a, p, x)$ denote the cumulative distribution function of $C - c$, the variable portion of the dual command travel time of a rack where a is the length of the high turnover zone and p is the probability that a request falls in the high turnover zone.

Theorem 2 *For* $0 \leq a < 1/2$,

$$
F(a,p,x) = \begin{cases}
F_1(x) & \textit{for } 0 \leq x < 2a, \\
F_2(x) & \textit{for } 2a \leq x < 3a, \\
F_3(x) & \textit{for } 3a \leq x < 4a, \\
F_4(x) & \textit{for } 4a \leq x < 2, \\
F_5(x) & \textit{for } 2 \leq x < 2 + a, \\
F_6(x) & \textit{for } 2 + a \leq x \leq 3.
\end{cases}
$$

For $1/2 \leq a < 2/3$,

$$
F(a,p,x) = \begin{cases}
F_1(x) & \textit{for } 0 \leq x < 2a, \\
F_2(x) & \textit{for } 2a \leq x < 3a, \\
F_3(x) & \textit{for } 3a \leq x < 2, \\
F_6(x) & \textit{for } 2 \leq x < 1 + 2a, \\
F_7(x) & \textit{for } 1 + 2a \leq x < 2 + a, \\
F_8(x) & \textit{for } 2 + a \leq x \leq 3.
\end{cases}
$$

For $2/3 \leq a \leq 1$,

$$F(a,p,x) = \begin{cases} F_1(x) & \textit{for } 0 \leq x < 2a, \\ F_2(x) & \textit{for } 2a \leq x < 2, \\ F_5(x) & \textit{for } 2 \leq x < 3a, \\ F_6(x) & \textit{for } 3a \leq x < 1+2a, \\ F_7(x) & \textit{for } 1+2a \leq x < 2+a, \\ F_8(x) & \textit{for } 2+a \leq x \leq 3. \end{cases}$$

The functions $F_1, \ldots, F_8$ *are defined in Appendix A.*

Proof The proof is quite similar to the proof of Theorem 2 in [4]. We wish to generate two independent, identically distributed random points over the unit square. The point should land with probability p in the high turnover zone and with probability $1-p$ in the low turnover zone. Given that a point falls in one of the zones, its location is uniformly distributed over that zone. It will be more convenient to first assume that the point falls on a certain curve, and then given that the point falls on that curve, determine the density function of its location on the curve. We select the curves $\{x \in [0,1]^2 : \|x\| = r\}$, for $0 \leq r \leq 1$. We will call the curve generated by a fixed value of r, curve r[1]. Curve r corresponds to the points on the two line segments joining $(0,r)$ with (r,r) and (r,r) with $(r,0)$ in Figure 2.

We select these curves for three reasons. First, they partition the unit square. Second, if the point falls on curve r, the point will be r units from the origin. Third, given the point lies on curve r, the location of the point is uniformly distributed over curve r.

The probability density function for the point landing on curve r in the uniform case is

$$g(r) = \begin{cases} 2rp/a^2 & \text{for } 0 < r < a \\ 2r(1-p)/(1-a^2) & \text{for } a \leq r < 1. \end{cases} \tag{1}$$

Since the two points, r and s, are independent, the joint probability density function of one point on curve r and the other on s is $g(r)g(s)$. If we let R be the shorter, S the longer leg, and $F_{r,s}(t)$ be the conditional distribution function for the distance between the two random points given that one is on curve r and the other on curve s, we have

$$F(a,p,x) = \int_0^1 \int_0^s F_{r,s} x - r - s\ 2!\ g(r)g(s)\, dr\, ds. \tag{2}$$

[1]The approach here is in the spirit of the approach used in the appendix to [7].

From [4],

$$F_{r,s}(t) = \begin{cases} F_A(t) & \text{if } t \geq s \\ F_B(t) & \text{if } t > s-r, r < t < s \\ F_C(t) & \text{if } t > s-r, t \leq r,\ 2t \leq s \\ F_D(t) & \text{if } s > t > s-r, t \leq r,\ 2t > s \\ 0 & \text{if } t \leq s-r \end{cases} \tag{3}$$

where

$$\begin{aligned}
F_A(t) &= 1 \\
F_B(t) &= \int_0^{(s-t)/r} \frac{ru+t}{2s} du + \frac{1}{2}\int_{(s-t)/r}^{1} du + \frac{1}{2}(1-\frac{s-t}{r}) \\
F_C(t) &= \int_0^{t/r} \frac{ru+t}{2s} du + \int_{t/r}^{(s-t)/r} \frac{t}{s} du + \int_{(s-t)/r}^{1} \frac{s-(ru-t)}{2s} du \\
&\quad + \frac{1}{2}(1-\frac{s-t}{r})(1-\frac{r-t}{s}) \\
F_D(t) &= \int_0^{(s-t)/r} \frac{ru+t}{2s} du + \frac{1}{2}\int_{(s-t)/r}^{t/r} du + \int_{t/r}^{1} \frac{s-(ru-t)}{2s} du \\
&\quad + \frac{1}{2}(1-\frac{s-t}{r})(1-\frac{r-t}{s}).
\end{aligned}$$

Substituting the expression for $F_{r,s}(t)$ and $g(\cdot)$ into (2) and integrating eventually yields the expressions given in Appendix A.

□

Corollary 2 *The mean dual command travel time for* $0 \leq a < 1/2$ *is*

$$\begin{aligned}
E[C-c] &= c + 3 - 10/3a^2c_1c_2 + 10/3a^2c_2^2 \\
&\quad + a^3c_1c_2/3 - a^3c_2^2/3 + 37/6a^4c_1c_2 \\
&\quad - 3a^4c_1^2 - 19/6a^4c_2^2 - 47/15a^5c_1c_2 \\
&\quad + 9/5a^5c_1^2 + 4/3a^5c_2^2 - 6/5c_2^2
\end{aligned}$$

and for $1/2 \leq a \leq 1$

$$\begin{aligned}
E[C-c] &= c + 3 - ac_1c_2/3 + ac_2^2/3 + c_1c_2/30 \\
&\quad - 2a^2c_1c_2 + 2a^2c_2^2 - 7/3a^3c_1c_2 \\
&\quad + 7/3a^3c_2^2 + 53/6a^4c_1c_2 - 3a^4c_1^2 \\
&\quad - 35/6a^4c_2^2 - 21/5a^5c_1c_2 + 9/5a^5c_1^2 \\
&\quad + 12/5a^5c_2^2 - 37/30c_2^2.
\end{aligned}$$

where

$$c_1 \equiv p/a^2 \text{ and}$$

$$c_2 \equiv (1-p)/(1-a^2)$$

Proof Given a, integrate the complementary distribution function corresponding to the distribution given in Theorem 2. The two cases $1/2 \leq a < 2/3$ and $2/3 \leq a \leq 1$ simplify to the same expression.

□

7 Throughput and Bounds

In Section 6, we derived the distribution of the dual command travel time for a storage rack with two activity zones. Given a particular pick time distribution, we can now derive expressions for the system throughput. The pick time distributions in Theorem 1 are particulary interesting since they also yield bounds for classes of distribution functions. Hence, we will determine the throughput for two of these distributions: exponential and deterministic.

Theorem 3 *Suppose that the pick times are exponentially distributed with mean $1/\lambda$. Then TPTis the expression given in Appendix expl. Suppose that the pick times are determinstic d. Then TPTis the expression given in Appendix det.*

Proof Simply compute $E[P \vee C]$ using the cumulative distribution function given in 2.

□

Corollary 3 *If the pick times are NBUE with mean m, then $\overline{TPT}$is bounded between the exponential with $1/\lambda = m$ and the deterministic with $d = m$.*

8 Examples

In this section, we give numerical examples of how the system throughput varies with zone size, pick time distribution, and activity skew. We assume that the s/r machine takes 1/2 minute to travel horizontally from the front of the rack to the back. To normalize the rack, we assume this length of time is 1, and all future times are in units of 1/2 minute.

Since the fraction of items designated as high-turnover increases as the size of the high turnover zone increases, we have assumed the following simple relationship. If the length of the high turnover zone is a, then a^2 of the items are stored in the high turnover zone. The amount of activity in the high turnover zone is

$$p = (a^2)^\alpha \tag{4}$$

. The parameter α determines how skewed the activity distribution is. If $\alpha = 1$, all items have the same activity and zoning will not affect throughput. Normally, $\alpha < 1$. If one point on the curve in (4) is known, the parameter α can be determined. For example if 80% of the activity is caused by only 20% of the items, then α is .139. In general, $\alpha = \log p / \log a^2$.

We have selected two activity skews: 80-20 and 90-10. These activity skews are reasonable in many applications, and we have seen cases where as much as 90% of the activity was concentrated on only 1% of the items. With such an extreme skew, it might be advisable to remove some of these items and store them in a more readily accessible location. We have set the fixed part of the dual command travel time to $c = 1/10$.

In Figure 3, the expected dual command travel time is plotted as a function of the high turnover zone size a. With $a = 0$ or $a = 1$, we have essentially no zoning. As a increases, the expected dual command travel time decreases and then increases to its original value. The optimal zone size for the 80-20 skew is larger than the optimal zone size for the 90-10 skew. The more skewed the distribution, the smaller the optimal zone size, and the more dramatic the decrease in expected dual command travel time. However, we need to include the pick time information to assess the effect of zoning on system performance.

Initially, we will consider both deterministic and exponential pick time distributions with a mean of 1. Figures 4 and 5, compare the throughput as a function of the zone size for the 80-20 and 90-10 skew, respectively. Note, the greater the skew, the smaller the optimal zone size, and the greater the improvement in throughput. In Figure 5, the increase in system throughput for optimal zoning over no zoning is particularly dramatic: 62% for the exponential pick times and 71% for the deterministic pick times.

In Figure 6, the mean pick time was increased from 1 to 1.8. Increasing the mean pick time, decreases throughput, and reduces the improvement available through zoning. When the activity skew is large and the s/r machine tends to be the bottleneck, zoning can dramatically increase throughput. As the pick time increases, the activity is unchanged. However, the picker becomes more of the bottleneck which reduces the impact of zoning as can be seen by comparing Figures 4 and 6.

In determining zone size, when the pick time distribution is unknown but the mean pick time is known and the pick time is NBUE, the safest procedure would be to use the optimal zone size from the exponential pick times. From Theorem 1, this policy guarantees the throughput will be at least as good as the throughput of the exponential case.

9 Summary

In this paper, we have considered the effect of activity zoning, pick time distribution, and activity skew on throughput and utilization. Specifically, we have derived bounds and closed form expressions for miniload system throughput, picker utilization, and s/r machine utilization.

These results have been implemented in a small software package which

Figure 3: Expected Dual Command Travel Time vs. Zonesize

Figure 4: Miniload Throughput vs. Zonesize

Figure 5: Miniload Throughput vs. Zonesize

Figure 6: Miniload Throughput vs. Zonesize

allows the designer to specify system parameters and determines throughput, resource utilization, and bounds on these quantities. The package also includes the rectangular rack without zoning which will be described in [13].

As a rule of thumb, the more skewed the activity distribution and the more the s/r machine tends to be the bottleneck, the more it pays to zone. In an example based on data from an existing system, zoning should increase throughput by 70%. The actual advantage of zoning will vary depending on system parameters, but can be estimated using the analytical results in this paper.

A Dual Command Travel Time Distribution with Two Zones

This appendix contains $F_1, \ldots, F_8$ which give the cumulative distribution function of the variable portion of the dual command travel time; see Theorem 2. Let c_1 be the height of the density function in the high turnover region. Thus, $c_1 \equiv p/a^2$. Similarly, let $c_2 \equiv (1-p)/(1-a^2)$.

$$\begin{aligned}
F_1(x) &\equiv 25/576x^4c_1^2 \\
F_2(x) &\equiv ax^3c_1c_2/12 - ax^3c_2^2/12 \\
&\quad +11/4a^2x^2c_1c_2 - 3/2a^2x^2c_1^2 - 5/4a^2x^2c_2^2 \\
&\quad -29/3a^3xc_1c_2 + 6a^3xc_1^2 + 11/3a^3xc_2^2 \\
&\quad +103/12a^4c_1c_2 - 23/4a^4c_1^2 - 17/6a^4c_2^2 \\
&\quad -11/192x^4c_1c_2 + x^4c_1^2/36 + 7/96x^4c_2^2 \\
F_3(x) &\equiv ax^3c_1c_2/4 - ax^3c_2^2/4 \\
&\quad -a^2x^2c_1c_2 + a^2x^2c_2^2 + 23/6a^3xc_1c_2 \\
&\quad -23/6a^3xc_2^2 - 145/24a^4c_1c_2 + a^4c_1^2 \\
&\quad +121/24a^4c_2^2 - x^4c_1c_2/64 + 17/288x^4c_2^2 \\
F_4(x) &\equiv a^2x^2c_1c_2/2 - a^2x^2c_2^2/2 \\
&\quad -a^3xc_1c_2/6 + a^3xc_2^2/6 - 49/24a^4c_1c_2 \\
&\quad +a^4c_1^2 + 25/24a^4c_2^2 + 25/576x^4c_2^2 \\
F_5(x) &\equiv 2axc_1c_2 - 2axc_2^2 \\
&\quad -10/3ac_1c_2 + 10/3ac_2^2 + 4xc_1c_2 + 2xc_2^2 \\
&\quad -35/12c_1c_2 + 2a^2xc_1c_2 - 2a^2xc_2^2 \\
&\quad +a^2c_1c_2 + 3/2a^2x^2c_1c_2 - 3/2a^2x^2c_1^2 \\
&\quad -a^2c_2^2 - 8a^3xc_1c_2 + 6a^3xc_1^2 + 2a^3xc_2^2 \\
&\quad -10/3a^3c_1c_2 + 10/3a^3c_2^2 + 103/12a^4c_1c_2 \\
&\quad -23/4a^4c_1^2 - 17/6a^4c_2^2 - 3/2x^2c_1c_2 \\
&\quad +x^4c_1^2/36 - 17/6c_2^2
\end{aligned}$$

$$
\begin{aligned}
F_6(x) \equiv\ & 2axc_1c_2 - 2axc_2^2 \\
& -10/3ac_1c_2 + ax^3c_1c_2/6 - ax^3c_2^2/6 \\
& +10/3ac_2^2 + 4xc_1c_2 + 2xc_2^2 - 35/12c_1c_2 \\
& +2a^2xc_1c_2 - 2a^2xc_2^2 + a^2c_1c_2 \\
& -9/4a^2x^2c_1c_2 + 9/4a^2x^2c_2^2 - a^2c_2^2 \\
& +11/2a^3xc_1c_2 - 11/2a^3xc_2^2 - 10/3a^3c_1c_2 \\
& +10/3a^3c_2^2 - 145/24a^4c_1c_2 + a^4c_1^2 \\
& +121/24a^4c_2^2 - 3/2x^2c_1c_2 + x^4c_1c_2/24 \\
& -x^4c_2^2/72 - 17/6c_2^2 \\
F_7(x) \equiv\ & 6axc_1c_2 - 6axc_2^2 - 16/3ac_1c_2 \\
& -2ax^2c_1c_2 + 2ax^2c_2^2 + ax^3c_1c_2/6 \\
& -ax^3c_2^2/6 + 16/3ac_2^2 + 16/3xc_1c_2 \\
& +2/3xc_2^2 - 10/3c_1c_2 + 2a^2xc_1c_2 \\
& -2a^2xc_2^2 - a^2c_1c_2 - a^2x^2c_1c_2/4 \\
& +a^2x^2c_2^2/4 + a^2c_2^2 + a^3xc_1c_2/6 \\
& -a^3xc_2^2/6 - 2/3a^3c_1c_2 + 2/3a^3c_2^2 \\
& -49/24a^4c_1c_2 + a^4c_1^2 + 25/24a^4c_2^2 \\
& -3x^2c_1c_2 + 3/2x^2c_2^2 + 2/3x^3c_1c_2 \\
& -2/3x^3c_2^2 - x^4c_1c_2/24 + 5/72x^4c_2^2 \\
& -29/12c_2^2 \\
F_8(x) \equiv\ & 6xc_2^2 + 2a^2c_1c_2 \\
& -2a^2c_2^2 - 2a^4c_1c_2 + a^4c_1^2 + a^4c_2^2 \\
& -3/2x^2c_2^2 + x^4c_2^2/36 - 23/4c_2^2
\end{aligned}
$$

B Throughput Equations for Deterministic Pick Times

This appendix contains closed form expressions for the throughput of a continuous, square-in-time rack with with two storage classes, and deterministic pick times. Although the expressions are lengthy and have several cases, the expressions are simply polynomials.

Recall that the rack has length one, p is the probability that a high turnover item is requested, a is the length of the high turnover portion of the rack, c is the constant part of the dual command travel time, and d is the deterministic pick time. Let c_1 be the height of the density function in the high turnover region. Thus, $c_1 \equiv p/a^2$. Similarly, let $c_2 \equiv (1-p)/(1-a^2)$. The throughput is

$1/E[P \vee C]$. For $a < 1/2$ and $0 \leq x < 2a$,

$$\begin{aligned} E[P \vee C] &= 3 + c - 10/3a^2c_1c_2 + 10/3a^2c_2^2 + a^3c_1c_2/3 \\ &\quad -a^3c_2^2/3 + 37/6a^4c_1c_2 - 3a^4c_1^2 - 19/6a^4c_2^2 - 47/15a^5c_1c_2 \\ &\quad +9/5a^5c_1^2 + 4/3a^5c_2^2 + 5/576c_1^2d^5 - 6/5c_2^2. \end{aligned}$$

For $a < 1/2$ and $2a \leq x < 3a$,

$$\begin{aligned} E[P \vee C] &= 3 + c + ac_1c_2d^4/48 - ac_2^2d^4/48 - 11/960c_1c_2d^5 \\ &\quad -10/3a^2c_1c_2 + 11/12a^2c_1c_2d^3 - a^2c_1^2d^3/2 + 10/3a^2c_2^2 - 5/12a^2c_2^2d^3 \\ &\quad +a^3c_1c_2/3 - 29/6a^3c_1c_2d^2 + 3a^3c_1^2d^2 - a^3c_2^2/3 + 11/6a^3c_2^2d^2 \\ &\quad +37/6a^4c_1c_2 + 103/12a^4c_1c_2d - 3a^4c_1^2 - 23/4a^4c_1^2d - 19/6a^4c_2^2 \\ &\quad -17/6a^4c_2^2d - 124/15a^5c_1c_2 + 27/5a^5c_1^2 + 43/15a^5c_2^2 + c_1^2d^5/180 \\ &\quad +7/480c_2^2d^5 - 6/5c_2^2 \end{aligned}$$

For $a < 1/2$ and $3a \leq x < 4a$,

$$\begin{aligned} E[P \vee C] &= 3 + c + ac_1c_2d^4/16 - ac_2^2d^4/16 - c_1c_2d^5/320 - 10/3a^2c_1c_2 \\ &\quad -a^2c_1c_2d^3/3 + 10/3a^2c_2^2 + a^2c_2^2d^3/3 + a^3c_1c_2/3 + 23/12a^3c_1c_2d^2 \\ &\quad -a^3c_2^2/3 - 23/12a^3c_2^2d^2 + 37/6a^4c_1c_2 - 145/24a^4c_1c_2d - 3a^4c_1^2 \\ &\quad +a^4c_1^2d - 19/6a^4c_2^2 + 121/24a^4c_2^2d + 77/24a^5c_1c_2 - 77/24a^5c_2^2 \\ &\quad +17/1440c_2^2d^5 - 6/5c_2^2 \end{aligned}$$

For $a < 1/2$ and $4a \leq x < 2$,

$$\begin{aligned} E[P \vee C] &= 3 + c - 10/3a^2c_1c_2 + a^2c_1c_2d^3/6 + 10/3a^2c_2^2 \\ &\quad -a^2c_2^2d^3/6 + a^3c_1c_2/3 - a^3c_1c_2d^2/12 - a^3c_2^2/3 + a^3c_2^2d^2/12 \\ &\quad +37/6a^4c_1c_2 - 49/24a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d - 19/6a^4c_2^2 \\ &\quad +25/24a^4c_2^2d + a^5c_1c_2/120 - a^5c_2^2/120 + 5/576c_2^2d^5 - 6/5c_2^2 \end{aligned}$$

For $a < 1/2$ and $2 \leq x < 2 + a$,

$$\begin{aligned} E[P \vee C] &= 3 + c + 10/3ac_1c_2 - 16/3ac_1c_2d + 3ac_1c_2d^2 \\ &\quad -2/3ac_1c_2d^3 + ac_1c_2d^4/24 - 10/3ac_2^2 + 16/3ac_2^2d - 3ac_2^2d^2 \\ &\quad +2/3ac_2^2d^3 - ac_2^2d^4/24 + 8/5c_1c_2 - 10/3c_1c_2d + 8/3c_1c_2d^2 \\ &\quad -c_1c_2d^3 + c_1c_2d^4/6 - c_1c_2d^5/120 - 10/3a^2c_1c_2 - a^2c_1c_2d \\ &\quad +a^2c_1c_2d^2 - a^2c_1c_2d^3/12 + 10/3a^2c_2^2 + a^2c_2^2d - a^2c_2^2d^2 \\ &\quad +a^2c_2^2d^3/12 + a^3c_1c_2 - 2/3a^3c_1c_2d + a^3c_1c_2d^2/12 - a^3c_2^2 \\ &\quad +2/3a^3c_2^2d - a^3c_2^2d^2/12 + 37/6a^4c_1c_2 - 49/24a^4c_1c_2d - 3a^4c_1^2 \\ &\quad +a^4c_1^2d - 19/6a^4c_2^2 + 25/24a^4c_2^2d + a^5c_1c_2/120 - a^5c_2^2/120 \\ &\quad -29/12c_2^2d + c_2^2d^2/3 + c_2^2d^3/2 - c_2^2d^4/6 + c_2^2d^5/72 + 4/5c_2^2 \end{aligned}$$

For $a < 1/2$ and $2 + a \leq x < 3$,

$$\begin{aligned} E[P \vee C] \;=\; & 3 + c - 6a^2c_1c_2 + 2a^2c_1c_2d + 6a^2c_2^2 \\ & -2a^2c_2^2d + 6a^4c_1c_2 - 2a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d \\ & -3a^4c_2^2 + a^4c_2^2d - 23/4c_2^2d + 3c_2^2d^2 - c_2^2d^3/2 \\ & +c_2^2d^5/180 + 12/5c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $0 \leq x < 2a$,

$$\begin{aligned} E[P \vee C] \;=\; & 3 + c - ac_1c_2/3 + ac_2^2/3 + c_1c_2/30 \\ & -2a^2c_1c_2 + 2a^2c_2^2 - 7/3a^3c_1c_2 + 7/3a^3c_2^2 + 53/6a^4c_1c_2 \\ & -3a^4c_1^2 - 35/6a^4c_2^2 - 21/5a^5c_1c_2 + 9/5a^5c_1^2 + 12/5a^5c_2^2 \\ & +5/576c_1^2d^5 - 37/30c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $2a \leq x < 3a$,

$$\begin{aligned} E[P \vee C] \;=\; & 3 + c - ac_1c_2/3 + ac_1c_2d^4/48 + ac_2^2/3 \\ & -ac_2^2d^4/48 + c_1c_2/30 - 11/960c_1c_2d^5 - 2a^2c_1c_2 + 11/12a^2c_1c_2d^3 \\ & -a^2c_1^2d^3/2 + 2a^2c_2^2 - 5/12a^2c_2^2d^3 - 7/3a^3c_1c_2 - 29/6a^3c_1c_2d^2 \\ & +3a^3c_1^2d^2 + 7/3a^3c_2^2 + 11/6a^3c_2^2d^2 + 53/6a^4c_1c_2 + 103/12a^4c_1c_2d \\ & -3a^4c_1^2 - 23/4a^4c_1^2d - 35/6a^4c_2^2 - 17/6a^4c_2^2d - 28/3a^5c_1c_2 \\ & +27/5a^5c_1^2 + 59/15a^5c_2^2 + c_1^2d^5/180 + 7/480c_2^2d^5 - 37/30c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $3a \leq x < 2$,

$$\begin{aligned} E[P \vee C] \;=\; & 3 + c - ac_1c_2/3 + ac_1c_2d^4/16 + ac_2^2/3 \\ & -ac_2^2d^4/16 + c_1c_2/30 - c_1c_2d^5/320 - 2a^2c_1c_2 - a^2c_1c_2d^3/3 \\ & +2a^2c_2^2 + a^2c_2^2d^3/3 - 7/3a^3c_1c_2 + 23/12a^3c_1c_2d^2 + 7/3a^3c_2^2 \\ & -23/12a^3c_2^2d^2 + 53/6a^4c_1c_2 - 145/24a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d \\ & -35/6a^4c_2^2 + 121/24a^4c_2^2d + 257/120a^5c_1c_2 - 257/120a^5c_2^2 + 17/1440c_2^2d^5 \\ & -37/30c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $2 \leq x < 1 + 2a$,

$$\begin{aligned} E[P \vee C] \;=\; & 3 + c + 8/3ac_1c_2 - 10/3ac_1c_2d + ac_1c_2d^2 \\ & +ac_1c_2d^4/24 - 8/3ac_2^2 + 10/3ac_2^2d - ac_2^2d^2 - ac_2^2d^4/24 \\ & +3/2c_1c_2 - 35/12c_1c_2d + 2c_1c_2d^2 - c_1c_2d^3/2 + c_1c_2d^5/120 \\ & -14/3a^2c_1c_2 + a^2c_1c_2d + a^2c_1c_2d^2 - 3/4a^2c_1c_2d^3 + 14/3a^2c_2^2 \\ & -a^2c_2^2d - a^2c_2^2d^2 + 3/4a^2c_2^2d^3 + a^3c_1c_2 - 10/3a^3c_1c_2d \\ & +11/4a^3c_1c_2d^2 - a^3c_2^2 + 10/3a^3c_2^2d - 11/4a^3c_2^2d^2 + 53/6a^4c_1c_2 \\ & -145/24a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d - 35/6a^4c_2^2 + 121/24a^4c_2^2d \\ & +257/120a^5c_1c_2 - 257/120a^5c_2^2 - 17/6c_2^2d + c_2^2d^2 - c_2^2d^5/360 \\ & +9/10c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $1 + 2a \leq x < 2 + a$,

$$\begin{aligned} E[P \vee C] &= 3 + c + 10/3ac_1c_2 - 16/3ac_1c_2d + 3ac_1c_2d^2 \\ &\quad -2/3ac_1c_2d^3 + ac_1c_2d^4/24 - 10/3ac_2^2 + 16/3ac_2^2d - 3ac_2^2d^2 \\ &\quad +2/3ac_2^2d^3 - ac_2^2d^4/24 + 8/5c_1c_2 - 10/3c_1c_2d + 8/3c_1c_2d^2 \\ &\quad -c_1c_2d^3 + c_1c_2d^4/6 - c_1c_2d^5/120 - 10/3a^2c_1c_2 - a^2c_1c_2d \\ &\quad +a^2c_1c_2d^2 - a^2c_1c_2d^3/12 + 10/3a^2c_2^2 + a^2c_2^2d - a^2c_2^2d^2 \\ &\quad +a^2c_2^2d^3/12 + a^3c_1c_2 - 2/3a^3c_1c_2d + a^3c_1c_2d^2/12 - a^3c_2^2 \\ &\quad +2/3a^3c_2^2d - a^3c_2^2d^2/12 + 37/6a^4c_1c_2 - 49/24a^4c_1c_2d - 3a^4c_1^2 \\ &\quad +a^4c_1^2d - 19/6a^4c_2^2 + 25/24a^4c_2^2d + a^5c_1c_2/120 - a^5c_2^2/120 \\ &\quad -29/12c_2^2d + c_2^2d^2/3 + c_2^2d^3/2 - c_2^2d^4/6 + c_2^2d^5/72 \\ &\quad +4/5c_2^2 \end{aligned}$$

For $1/2 \leq a < 2/3$ and $2 + a \leq x < 3$,

$$\begin{aligned} E[P \vee C] &= 3 + c - 6a^2c_1c_2 + 2a^2c_1c_2d + 6a^2c_2^2 \\ &\quad -2a^2c_2^2d + 6a^4c_1c_2 - 2a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d \\ &\quad -3a^4c_2^2 + a^4c_2^2d - 23/4c_2^2d + 3c_2^2d^2 - c_2^2d^3/2 \\ &\quad +c_2^2d^5/180 + 12/5c_2^2 \end{aligned}$$

For $2/3 \leq a \leq 1$ and $0 \leq x < 2a$,

$$\begin{aligned} E[P \vee C] &= 3 + c - ac_1c_2/3 + ac_2^2/3 + c_1c_2/30 \\ &\quad -2a^2c_1c_2 + 2a^2c_2^2 - 7/3a^3c_1c_2 + 7/3a^3c_2^2 + 53/6a^4c_1c_2 \\ &\quad -3a^4c_1^2 - 35/6a^4c_2^2 - 21/5a^5c_1c_2 + 9/5a^5c_1^2 + 12/5a^5c_2^2 \\ &\quad +5/576c_1^2d^5 - 37/30c_2^2 \end{aligned}$$

For $2/3 \leq a \leq 1$ and $2a \leq x < 2$,

$$\begin{aligned} E[P \vee C] &= 3 + c - ac_1c_2/3 + ac_1c_2d^4/48 + ac_2^2/3 \\ &\quad -ac_2^2d^4/48 + c_1c_2/30 - 11/960c_1c_2d^5 - 2a^2c_1c_2 + 11/12a^2c_1c_2d^3 \\ &\quad -a^2c_1^2d^3/2 + 2a^2c_2^2 - 5/12a^2c_2^2d^3 - 7/3a^3c_1c_2 - 29/6a^3c_1c_2d^2 \\ &\quad +3a^3c_1^2d^2 + 7/3a^3c_2^2 + 11/6a^3c_2^2d^2 + 53/6a^4c_1c_2 + 103/12a^4c_1c_2d \\ &\quad -3a^4c_1^2 - 23/4a^4c_1^2d - 35/6a^4c_2^2 - 17/6a^4c_2^2d - 28/3a^5c_1c_2 \\ &\quad +27/5a^5c_1^2 + 59/15a^5c_2^2 + c_1^2d^5/180 + 7/480c_2^2d^5 - 37/30c_2^2 \end{aligned}$$

For $2/3 \leq a \leq 1$ and $2 \leq x < 3a$,

$$\begin{aligned} E[P \vee C] &= 3 + c + 8/3ac_1c_2 - 10/3ac_1c_2d + ac_1c_2d^2 \\ &\quad -8/3ac_2^2 + 10/3ac_2^2d - ac_2^2d^2 + 3/2c_1c_2 - 35/12c_1c_2d \end{aligned}$$

$$+2c_1c_2d^2 - c_1c_2d^3/2 - 14/3a^2c_1c_2 + a^2c_1c_2d + a^2c_1c_2d^2$$
$$+a^2c_1c_2d^3/2 - a^2c_1^2d^3/2 + 14/3a^2c_2^2 - a^2c_2^2d - a^2c_2^2d^2$$
$$+a^3c_1c_2 - 10/3a^3c_1c_2d - 4a^3c_1c_2d^2 + 3a^3c_1^2d^2 - a^3c_2^2$$
$$+10/3a^3c_2^2d + a^3c_2^2d^2 + 53/6a^4c_1c_2 + 103/12a^4c_1c_2d - 3a^4c_1^2$$
$$-23/4a^4c_1^2d - 35/6a^4c_2^2 - 17/6a^4c_2^2d - 28/3a^5c_1c_2 + 27/5a^5c_1^2$$
$$+59/15a^5c_2^2 + c_1^2d^5/180 - 17/6c_2^2d + c_2^2d^2 + 9/10c_2^2$$

For $2/3 \leq a \leq 1$ and $3a \leq x < 1 + 2a$,

$$\begin{aligned}
E[P \vee C] = \; & 3 + c + 8/3ac_1c_2 - 10/3ac_1c_2d + ac_1c_2d^2 \\
& +ac_1c_2d^4/24 - 8/3ac_2^2 + 10/3ac_2^2d - ac_2^2d^2 - ac_2^2d^4/24 \\
& +3/2c_1c_2 - 35/12c_1c_2d + 2c_1c_2d^2 - c_1c_2d^3/2 + c_1c_2d^5/120 \\
& -14/3a^2c_1c_2 + a^2c_1c_2d + a^2c_1c_2d^2 - 3/4a^2c_1c_2d^3 + 14/3a^2c_2^2 \\
& -a^2c_2^2d - a^2c_2^2d^2 + 3/4a^2c_2^2d^3 + a^3c_1c_2 - 10/3a^3c_1c_2d \\
& +11/4a^3c_1c_2d^2 - a^3c_2^2 + 10/3a^3c_2^2d - 11/4a^3c_2^2d^2 + 53/6a^4c_1c_2 \\
& -145/24a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d - 35/6a^4c_2^2 + 121/24a^4c_2^2d \\
& +257/120a^5c_1c_2 - 257/120a^5c_2^2 - 17/6c_2^2d + c_2^2d^2 - c_2^2d^5/360 \\
& +9/10c_2^2
\end{aligned}$$

For $2/3 \leq a \leq 1$ and $1 + 2a \leq x < 2 + a$,

$$\begin{aligned}
E[P \vee C] = \; & 3 + c + 10/3ac_1c_2 - 16/3ac_1c_2d + 3ac_1c_2d^2 \\
& -2/3ac_1c_2d^3 + ac_1c_2d^4/24 - 10/3ac_2^2 + 16/3ac_2^2d - 3ac_2^2d^2 \\
& +2/3ac_2^2d^3 - ac_2^2d^4/24 + 8/5c_1c_2 - 10/3c_1c_2d + 8/3c_1c_2d^2 \\
& -c_1c_2d^3 + c_1c_2d^4/6 - c_1c_2d^5/120 - 10/3a^2c_1c_2 - a^2c_1c_2d \\
& +a^2c_1c_2d^2 - a^2c_1c_2d^3/12 + 10/3a^2c_2^2 + a^2c_2^2d - a^2c_2^2d^2 \\
& +a^2c_2^2d^3/12 + a^3c_1c_2 - 2/3a^3c_1c_2d + a^3c_1c_2d^2/12 - a^3c_2^2 \\
& +2/3a^3c_2^2d - a^3c_2^2d^2/12 + 37/6a^4c_1c_2 - 49/24a^4c_1c_2d - 3a^4c_1^2 \\
& +a^4c_1^2d - 19/6a^4c_2^2 + 25/24a^4c_2^2d + a^5c_1c_2/120 - a^5c_2^2/120 \\
& -29/12c_2^2d + c_2^2d^2/3 + c_2^2d^3/2 - c_2^2d^4/6 + c_2^2d^5/72 \\
& +4/5c_2^2
\end{aligned}$$

For $2/3 \leq a \leq 1$ and $2 + a \leq x < 3$,

$$\begin{aligned}
E[P \vee C] = \; & 3 + c - 6a^2c_1c_2 + 2a^2c_1c_2d + 6a^2c_2^2 \\
& -2a^2c_2^2d + 6a^4c_1c_2 - 2a^4c_1c_2d - 3a^4c_1^2 + a^4c_1^2d \\
& -3a^4c_2^2 + a^4c_2^2d - 23/4c_2^2d + 3c_2^2d^2 - c_2^2d^3/2 \\
& +c_2^2d^5/180 + 12/5c_2^2
\end{aligned}$$

C Throughput Equations for Exponential Pick Times

This appendix contains closed form expressions for the throughput of a continuous, square-in-time rack with with two storage classes, and exponential pick times.

Recall that the rack has length one, p is the probability that a high turnover item is requested, a is the length of the high turnover portion of the rack, c is the constant part of the dual command travel time, and $1/\lambda$ is the mean pick time. Let c_1 be the height of the density function in the high turnover region. Thus, $c_1 \equiv p/a^2$. Similarly, let $c_2 \equiv (1-p)/(1-a^2)$. Let $L = -\lambda$. The throughput is $1/E[P \vee C]$.

For $a < 1/2$,

$$\begin{aligned}
E[P \vee C] &= 3 + c - 9/4a\exp(2aL + cL)c_1c_2/L^4 - 3/4a\exp(2aL + cL)c_1^2/L^4 \\
&\quad +3a\exp(2aL + cL)c_2^2/L^4 + a\exp(2L + cL)c_1c_2/L^4 \\
&\quad +2a\exp(2L + cL)c_1c_2/L^3 - a\exp(2L + cL)c_2^2/L^4 \\
&\quad -2a\exp(2L + cL)c_2^2/L^3 + 4a\exp(3aL + cL)c_1c_2/L^4 \\
&\quad -2a\exp(3aL + cL)c_1^2/L^4 - 2a\exp(3aL + cL)c_2^2/L^4 \\
&\quad -10/3a^2c_1c_2 - 15/4a^2\exp(2aL + cL)c_1c_2/L^3 \\
&\quad +15/4a^2\exp(2aL + cL)c_1^2/L^3 + 3/2a^2\exp(2L + cL)c_1c_2/L^3 \\
&\quad -a^2\exp(2L + cL)c_1c_2/L^2 - 3/2a^2\exp(2L + cL)c_2^2/L^3 \\
&\quad +a^2\exp(2L + cL)c_2^2/L^2 + 2a^2\exp(3L + cL)c_1c_2/L \\
&\quad -2a^2\exp(3L + cL)c_2^2/L + 10/3a^2c_2^2 + a^3c_1c_2/3 \\
&\quad +a^3\exp(2aL + cL)c_1c_2/(2L^2) - a^3\exp(2aL + cL)c_1^2/(2L^2) \\
&\quad +a^3\exp(2L + cL)c_1c_2/(3L^2) - a^3\exp(2L + cL)c_2^2/(3L^2) \\
&\quad -a^3c_2^2/3 + 37/6a^4c_1c_2 - 2a^4\exp(3L + cL)c_1c_2/L \\
&\quad +a^4\exp(3L + cL)c_1^2/L + a^4\exp(3L + cL)c_2^2/L \\
&\quad -3a^4c_1^2 - 19/6a^4c_2^2 - 47/15a^5c_1c_2 \\
&\quad +9/5a^5c_1^2 + 4/3a^5c_2^2 + 11/8\exp(2aL + cL)c_1c_2/L^5 \\
&\quad +3/8\exp(2aL + cL)c_1^2/L^5 - 7/4\exp(2aL + cL)c_2^2/L^5 \\
&\quad -\exp(2L + aL + cL)c_1c_2/L^5 - 2\exp(2L + aL + cL)c_1c_2/L^4 \\
&\quad +\exp(2L + aL + cL)c_2^2/L^5 + 2\exp(2L + aL + cL)c_2^2/L^4 \\
&\quad +\exp(2L + cL)c_1c_2/L^5 + 2\exp(2L + cL)c_1c_2/L^4 \\
&\quad -5/8\exp(2L + cL)c_2^2/L^5 - 11/4\exp(2L + cL)c_2^2/L^4 \\
&\quad +15/4\exp(2L + cL)c_2^2/L^3 - \exp(2L + cL)c_2^2/(2L^2) \\
&\quad -\exp(3aL + cL)c_1c_2/L^5 + 2/3\exp(3aL + cL)c_1^2/L^5 \\
&\quad +\exp(3aL + cL)c_2^2/(3L^5) + 2/3\exp(3L + cL)c_2^2/L^5
\end{aligned}$$

$$-2\exp(3L+cL)c_2^2/L^4+\exp(3L+cL)c_2^2/L$$
$$-\exp(3L+cL)/L-3/8\exp(4aL+cL)c_1c_2/L^5$$
$$+3/8\exp(4aL+cL)c_2^2/L^5-25/24\exp(cL)c_1^2/L^5$$
$$-6/5c_2^2$$

For $1/2 \le a < 1$,

$$\begin{aligned}
E[P\vee C] \;=\; & 3+c-ac_1c_2/3-4a\exp(L+2aL+cL)c_1c_2/L^4 \\
& +4a\exp(L+2aL+cL)c_2^2/L^4-9/4a\exp(2aL+cL)c_1c_2/L^4 \\
& -3/4a\exp(2aL+cL)c_1^2/L^4+3a\exp(2aL+cL)c_2^2/L^4 \\
& -a\exp(2L+cL)c_1c_2/(2L^4)+a\exp(2L+cL)c_1c_2/L^3 \\
& +a\exp(2L+cL)c_1c_2/L^2+a\exp(2L+cL)c_2^2/(2L^4) \\
& -a\exp(2L+cL)c_2^2/L^3-a\exp(2L+cL)c_2^2/L^2 \\
& +4a\exp(3aL+cL)c_1c_2/L^4-2a\exp(3aL+cL)c_1^2/L^4 \\
& -2a\exp(3aL+cL)c_2^2/L^4+ac_2^2/3+c_1c_2/30-2a^2c_1c_2 \\
& -15/4a^2\exp(2aL+cL)c_1c_2/L^3+15/4a^2\exp(2aL+cL)c_1^2/L^3 \\
& +5/2a^2\exp(2L+cL)c_1c_2/L^3-3a^2\exp(2L+cL)c_1c_2/L^2 \\
& -5/2a^2\exp(2L+cL)c_2^2/L^3+3a^2\exp(2L+cL)c_2^2/L^2 \\
& +2a^2\exp(3L+cL)c_1c_2/L-2a^2\exp(3L+cL)c_2^2/L \\
& +2a^2c_2^2-7/3a^3c_1c_2+a^3\exp(2aL+cL)c_1c_2/(2L^2) \\
& -a^3\exp(2aL+cL)c_1^2/(2L^2)+5/3a^3\exp(2L+cL)\overline{c_1}c_2/L^2 \\
& -5/3a^3\exp(2L+cL)c_2^2/L^2+7/3a^3c_2^2+53/6a^4c_1c_2 \\
& -2a^4\exp(3L+cL)c_1c_2/L+a^4\exp(3L+cL)c_1^2/L \\
& +a^4\exp(3L+cL)c_2^2/L-3a^4c_1^2-35/6a^4c_2^2 \\
& -21/5a^5c_1c_2+9/5a^5c_1^2+12/5a^5c_2^2 \\
& +2\exp(L+2aL+cL)c_1c_2/L^5+2\exp(L+2aL+cL)c_1c_2/L^4 \\
& -2\exp(L+2aL+cL)c_2^2/L^5-2\exp(L+2aL+cL)c_2^2/L^4 \\
& +11/8\exp(2aL+cL)c_1c_2/L^5+3/8\exp(2aL+cL)c_1^2/L^5 \\
& -7/4\exp(2aL+cL)c_2^2/L^5-\exp(2L+aL+cL)c_1c_2/L^5 \\
& -2\exp(2L+aL+cL)c_1c_2/L^4+\exp(2L+aL+cL)c_2^2/L^5 \\
& +2\exp(2L+aL+cL)c_2^2/L^4-11/8\exp(2L+cL)c_1c_2/L^5 \\
& +11/4\exp(2L+cL)c_1c_2/L^4+\exp(2L+cL)c_1c_2/(4L^3) \\
& -\exp(2L+cL)c_1c_2/(6L^2)+7/4\exp(2L+cL)c_2^2/L^5 \\
& -7/2\exp(2L+cL)c_2^2/L^4+7/2\exp(2L+cL)c_2^2/L^3 \\
& -\exp(2L+cL)c_2^2/(3L^2)-\exp(3aL+cL)c_1c_2/L^5 \\
& +2/3\exp(3aL+cL)c_1^2/L^5+\exp(3aL+cL)c_2^2/(3L^5) \\
& +2/3\exp(3L+cL)c_2^2/L^5-2\exp(3L+cL)c_2^2/L^4
\end{aligned}$$

$$+\exp(3L + cL)c_2^2/L - \exp(3L + cL)/L$$
$$-25/24\exp(cL)c_1^2/L^5 - 37/30c_2^2$$

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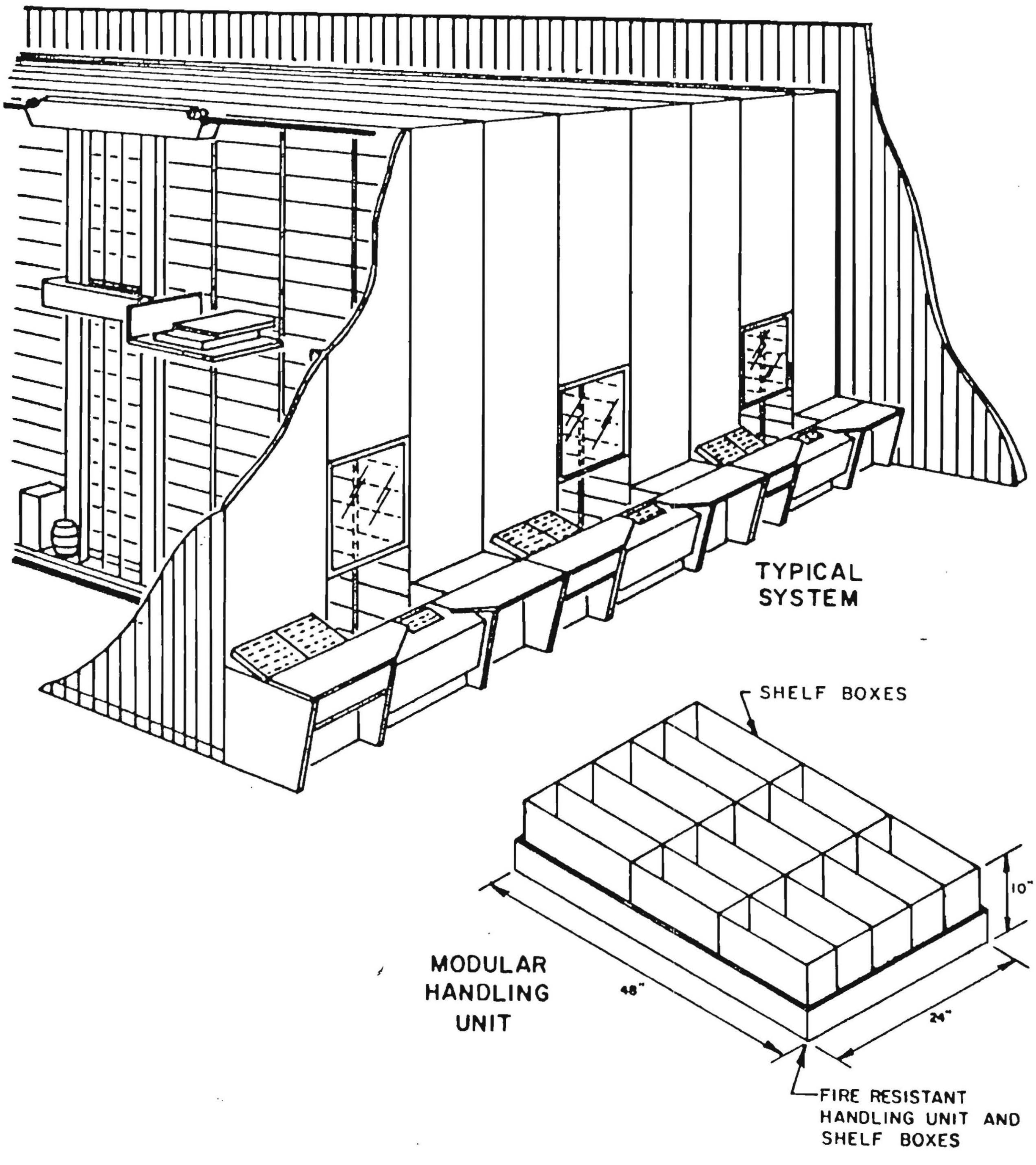
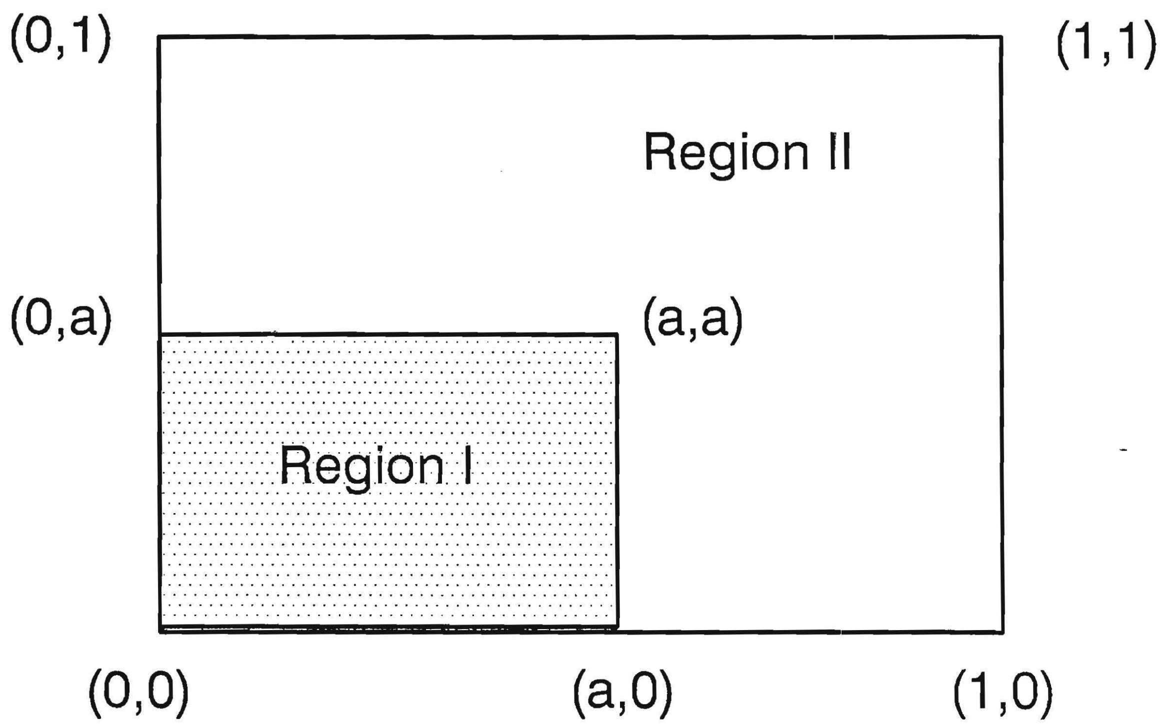
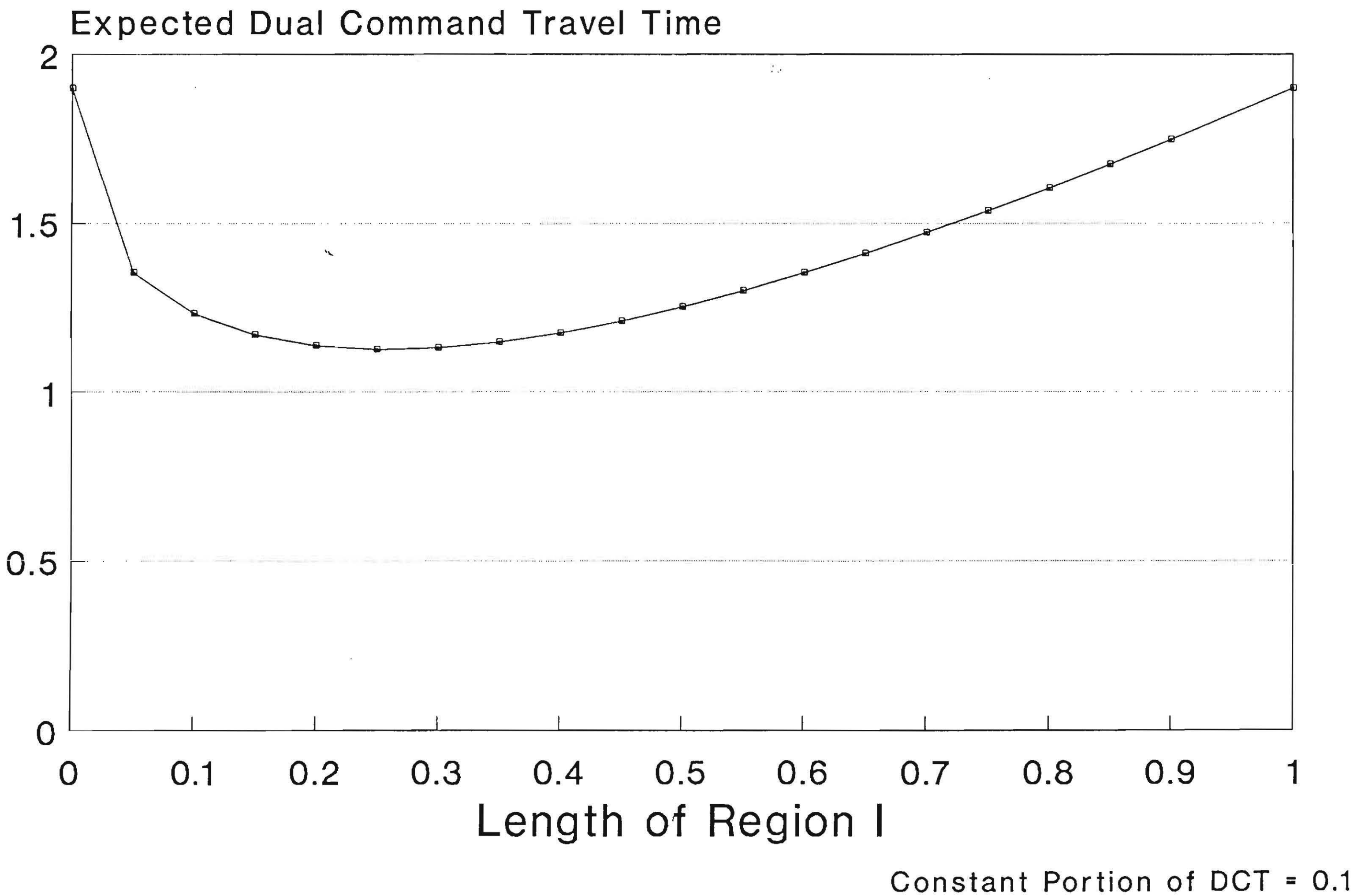


TYPICAL
MINI - S/R MACHINE

8198

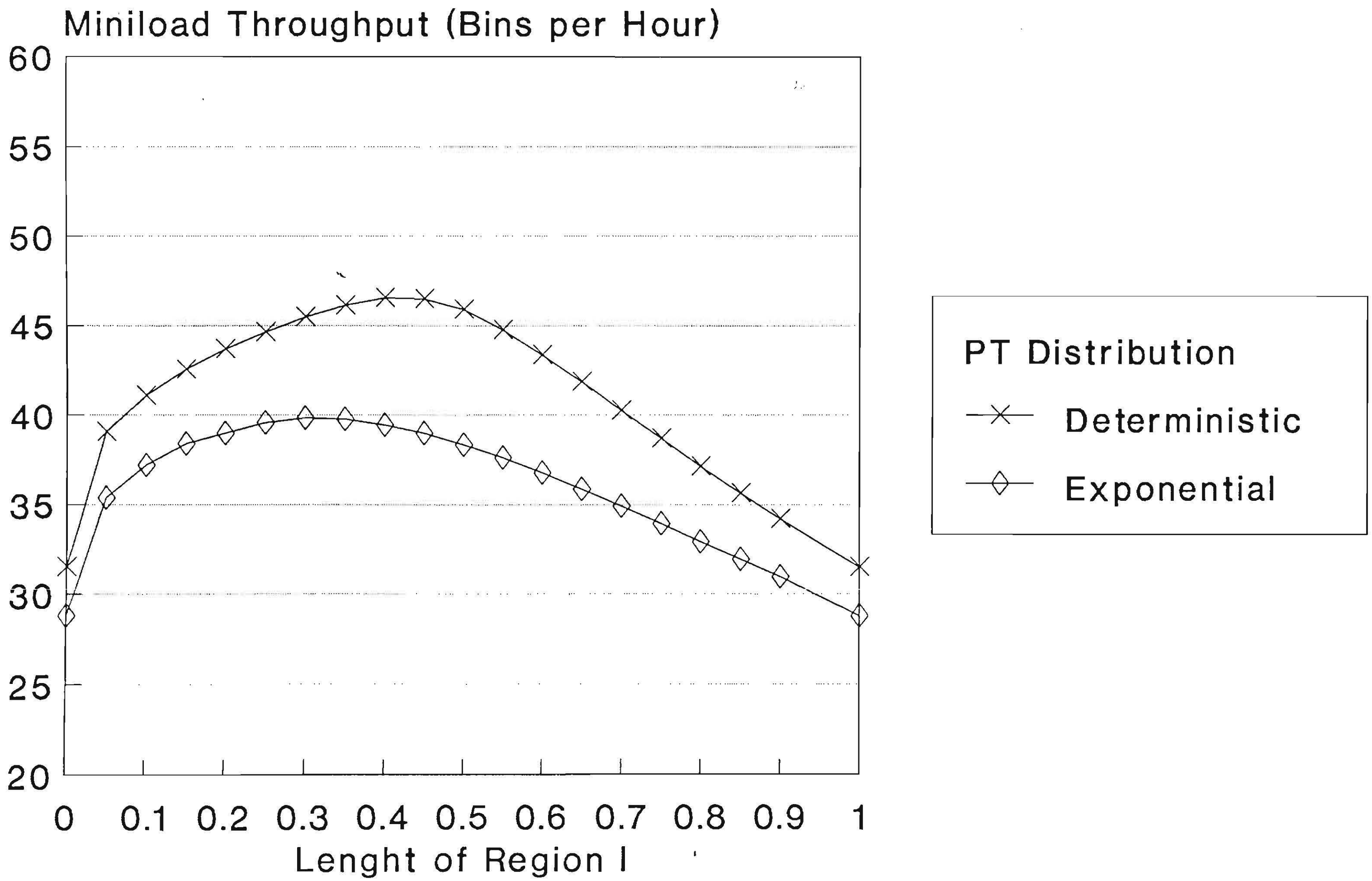

(0,1)
(1,1)
Region II
(0,a)
(a,a)
Region I
(0,0)
(a,0)
(1,0)

Expected Dual Command Travel vs. Region I Size



Miniload Throughput vs. Region I Size

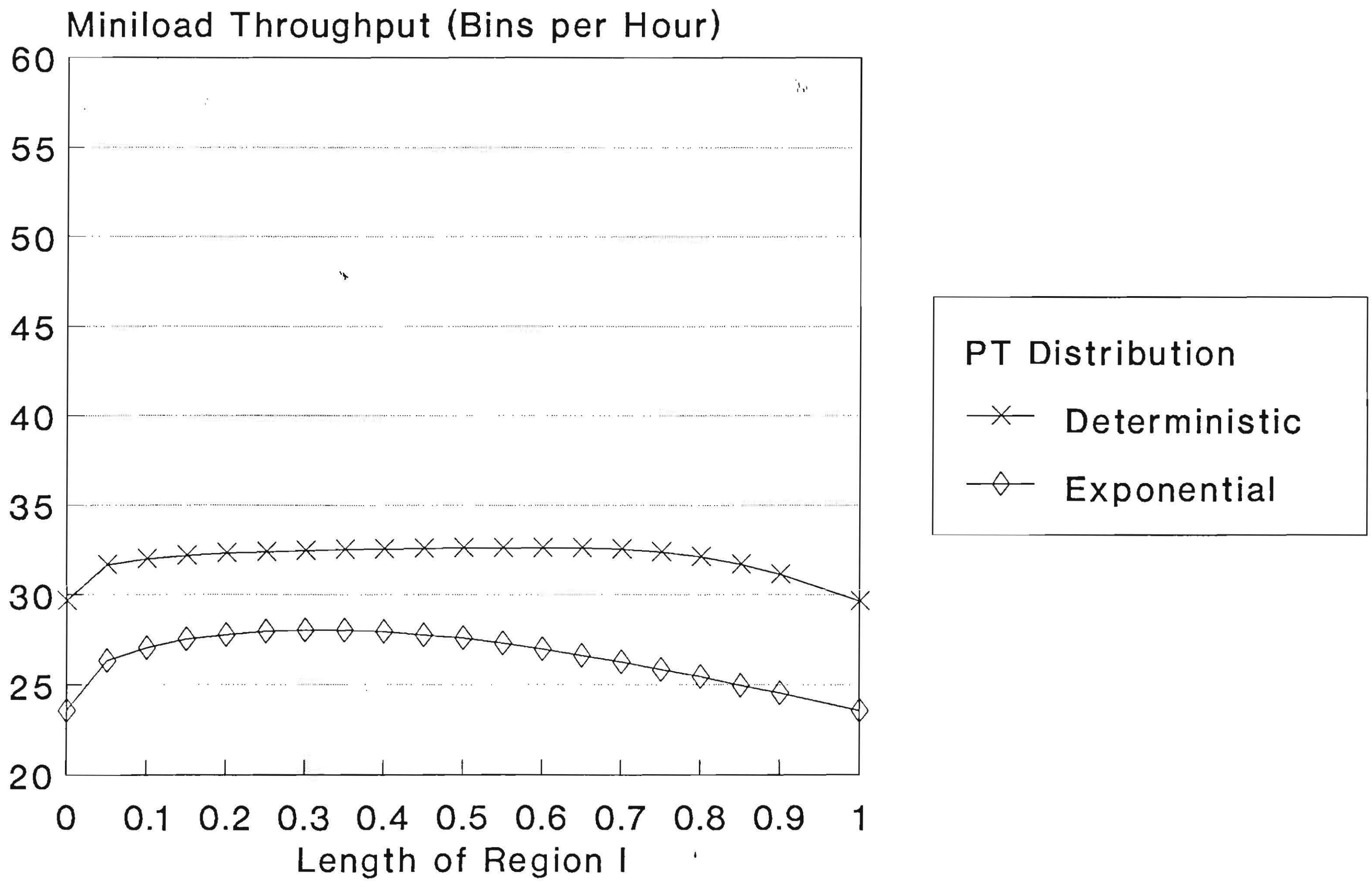
20% of Items Generate 80% of Retrievals



Mean pick time is 1.0 minutes per bin

Miniload Throughput vs. Region I Size

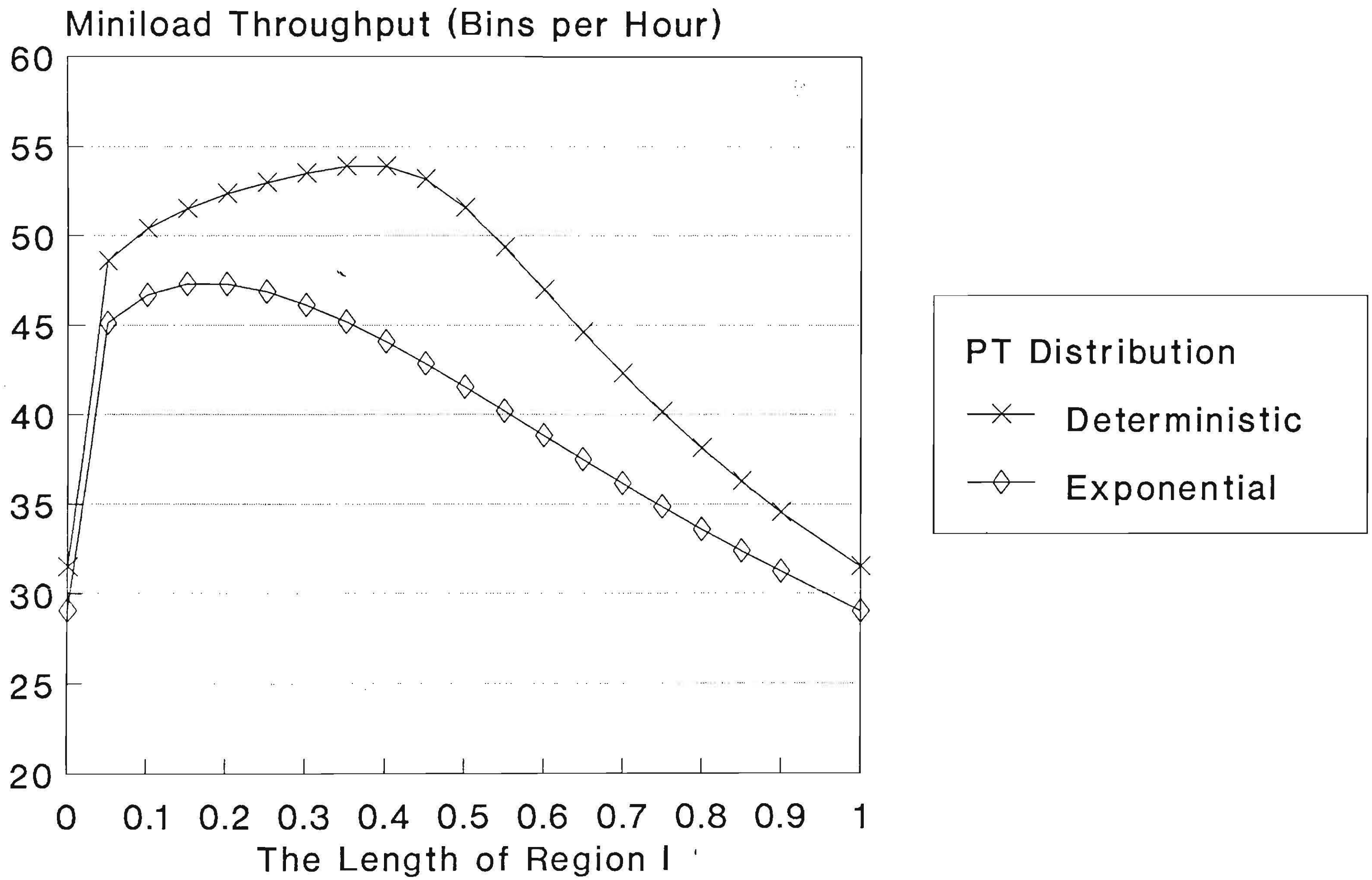
20% of Items Generate 80% of Retrievals



Mean pick time is 1.8 minutes per bin

Miniload Throughput vs. Region I Size

10% of Items Generate 90% of Retrievals



Mean pick time is 1.0 minutes per bin

E24-672

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT
NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Tech Research Center Georgia Institute of Technology Atlanta, Georgia 30332	Decision, Risk, Mgmt Sci	SES - 8821999
	4. Award Period From 5/1/89 To 10/31/91	5. Cumulative Award Amount $133,700

6. Project Title

ANALYTICAL RESULTS FOR MATERIAL HANDLING SYSTEMS

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

We developed useful analytical tools for modelling and improving the design of several material handling systems including a variety of miniload and carousel systems. Some of the results were validated by collecting data from existing industrial systems and comparing their performance with the performance predicted by our analytical results.

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM	
				Check (✓)	Approx. Date
a. Abstracts of Theses				X	15 Sept. 1992
b. Publication Citations				X	15 Sept. 1992
c. Data on Scientific Collaborators		X			
d. Information on Inventions	X				
e. Technical Description of Project and Results					
f. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Robert D. Foley		7/29/92

PART IV - SUMMARY DATA ON PROJECT PERSONNEL

NSF Division Social and Economic Sciences

The data requested below will be used to develop a statistical profile on the personnel supported through NSF grants. The information on this part is solicited under the authority of the National Science Foundation Act of 1950, as amended. All information provided will be treated as confidential and will be safeguarded in accordance with the provisions of the Privacy Act of 1974. NSF requires that a single copy of this part be submitted with each Final Project Report (NSF Form 98A); however, submission of the requested information is not mandatory and is not a precondition of future awards. If you do not wish to submit this information, please check this box ☐

Please enter the numbers of individuals supported under this NSF grant.
Do not enter information for individuals working less than 40 hours in any calendar year.

*U.S. Citizens/ Permanent Visa	PI's/PD's		Post-doctorals		Graduate Students		Under-graduates		Precollege Teachers		Others	
	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.
American Indian or Alaskan Native												
Asian or Pacific Islander												
Black, Not of Hispanic Origin												
Hispanic												
White, Not of Hispanic Origin	2				4							
Total U.S. Citizens	2				1							
Non U.S. Citizens					3							
Total U.S. & Non-U.S. ..	2				4							
Number of individuals who have a handicap that limits a major life activity.												

*Use the category that best describes person's ethnic/racial status. (If more than one category applies, use the one category that most closely reflects the person's recognition in the community.)

AMERICAN INDIAN OR ALASKAN NATIVE: A person having origins in any of the original peoples of North America, and who maintains cultural identification through tribal affiliation or community recognition.

ASIAN OR PACIFIC ISLANDER: A person having origins in any of the original peoples of the Far East, Southeast Asia, the Indian subcontinent, or the Pacific Islands. This area includes, for example, China, India, Japan, Korea, the Philippine Islands and Samoa.

BLACK, NOT OF HISPANIC ORIGIN: A person having origins in any of the black racial groups of Africa.

HISPANIC: A person of Mexican, Puerto Rican, Cuban, Central or South American or other Spanish culture or origin, regardless of race.

WHITE, NOT OF HISPANIC ORIGIN: A person having origins in any of the original peoples of Europe, North Africa or the Middle East.

THIS PART WILL BE PHYSICALLY SEPARATED FROM THE FINAL PROJECT REPORT AND USED AS A COMPUTER SOURCE DOCUMENT. DO NOT DUPLICATE IT ON THE REVERSE OF ANY OTHER PART OF THE FINAL REPORT.